Algebra & Precalculus Review

Math 135: Calculus for the Life Sciences — Brief Lecture Notes

1 Functions

Definition (function)

A **function** f is a rule that assigns to each input x *exactly one* output $f(x)$.

- Think input $\to$ machine $\to$ output. Writing $y = f(x)$: x is the independent variable, y the dependent variable.
- "Exactly one output" is the whole definition. A circle $x^2 + y^2 = 1$ is *not* the graph of a function ($x = 0$ gives $y = \pm 1$); graphically this is the **vertical line test**: every vertical line hits the graph of a function at most once.
- Life-science functions are everywhere: $N(t)$ = number of bacteria at time t; $C(d)$ = drug concentration in the blood at dose d. Same input measured twice can't give two different bacteria counts — that's why models are functions.

Example

$f(x) = x^2 + 1$ is a function: each x gives one value. $f(2) = 5$, $f(-2) = 5$ — two inputs may share an output; that's allowed. What is never allowed is one input with two outputs.

2 Domain and Range

Definition (domain and range)

The **domain** of f is the set of all allowed inputs. The **range** is the set of all outputs actually produced.

- Unless told otherwise, the domain is every real x for which the formula makes sense. The three usual restrictions:
  - denominators: cannot be 0;
  - even roots: $\sqrt{\text{stuff}}$ needs stuff ≥ 0;
  - logarithms: $\ln(\text{stuff})$ needs stuff > 0 (strict!).
- In applications the *context* can shrink the domain further (a width W must satisfy $W > 0$ even if the formula allows more).

Examples

1. $f(x) = \dfrac{1}{x-3}$: need $x \neq 3$. Domain $(-\infty, 3) \cup (3, \infty)$. Range: all $y \neq 0$.
2. $g(x) = \sqrt{x-2}$: need $x - 2 \geq 0$. Domain $[2, \infty)$; range $[0, \infty)$.
3. $h(x) = \dfrac{\sqrt{x}}{\ln(x)}$: need $x \geq 0$, $x > 0$, and $\ln(x) \neq 0$, i.e. $x \neq 1$. Domain $(0, 1) \cup (1, \infty)$.
4. $f(x) = x^2 + 1$: domain $(-\infty, \infty)$; since $x^2 \geq 0$, the range is $[1, \infty)$.

3 Even and Odd Functions

Definition (even, odd)

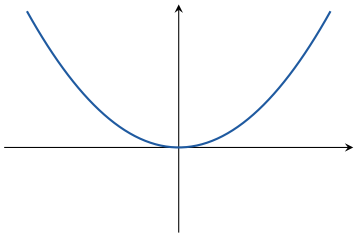
f is **even** if $f(-x) = f(x)$ for every x in the domain (graph symmetric about the y-axis).
f is **odd** if $f(-x) = -f(x)$ for every x in the domain (graph symmetric through the origin).

- Test by computing $f(-x)$ and simplifying — compare to $f(x)$ and to $-f(x)$.
- Even: x^2, x^4, $|x|$, $\cos(x)$. Odd: x, x^3, $1/x$, $\sin(x)$, $\tan(x)$.
- Most functions are *neither* (e.g. e^x, $\ln x$, $x^2 + x$).

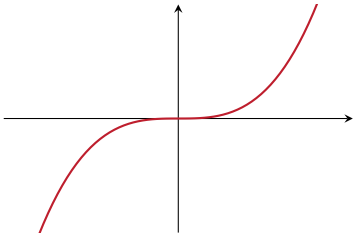
Examples

1. $f(x) = 3x^4 - 5x^2 + 2$: $f(-x) = 3x^4 - 5x^2 + 2 = f(x)$ $\Rightarrow$ even. (Only even powers appear.)
2. $g(x) = x^3 - 4x$: $g(-x) = -x^3 + 4x = -(x^3 - 4x) = -g(x)$ $\Rightarrow$ odd. (Only odd powers.)
3. $h(x) = x^2 + x$: $h(-x) = x^2 - x$, which is neither $h(x)$ nor $-h(x)$ $\Rightarrow$ neither.

even: $y = x^2$



odd: $y = x^3$



4 Secant Lines and Average Rate of Change

Definition (secant line)

A **secant line** is any line connecting two points on a curve. Its slope through $(a, f(a))$ and $(b, f(b))$ is

$$m_{\text{sec}} = \frac{f(b) - f(a)}{b - a} \qquad (= \text{the } \textbf{average rate of change} \text{ of } f \text{ on } [a, b]).$$

- "Slope = rise over run" applied to two points of the graph; units are (output units) per (input unit).
- In calculus we will shrink $[a, b]$ to a point: secant slopes $\to$ tangent slope = the derivative. This formula is the single most important prerequisite.

Example

A bacteria population is $N(t) = t^2 + t$ (thousands) at hour t. On $[1, 4]$:

$$m_{\text{sec}} = \frac{N(4) - N(1)}{4 - 1} = \frac{20 - 2}{3} = 6,$$

so the population grew by an average of 6,000 bacteria per hour between hours 1 and 4 — the slope of the line through $(1, 2)$ and $(4, 20)$.

Example (difference quotient)

For $f(x) = x^2$, the secant slope through x and $x + h$ ($h \neq 0$):

$$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{h(2x + h)}{h} = 2x + h.$$

Simplifying until the h in the denominator cancels is exactly the algebra the derivative will require.

5 Algebra Simplification Toolkit

Key facts

- **Exponent rules:** $x^a x^b = x^{a+b}$, $\dfrac{x^a}{x^b} = x^{a-b}$, $(x^a)^b = x^{ab}$, $x^{-a} = \dfrac{1}{x^a}$, $x^{1/n} = \sqrt[n]{x}$.
- **Factor out the smallest power**, including negative/fractional ones.
- **Fractions:** combine over a common denominator; to divide, multiply by the reciprocal.
- **Warning (the classic errors):** $(x + y)^2 \neq x^2 + y^2$, $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$, you may only cancel *factors*, never terms.

Examples

1. Factor completely: $5y^2(y-3)^5+10y(y-3)^4 = 5y(y-3)^4\big[y(y-3)+2\big] = 5y(y-3)^4(y^2-3y+2)$ $= 5y(y-3)^4(y-1)(y-2)$.

2. Factor (assume $x>0$): $3x^{-1/2}+4x^{1/2}+x^{3/2} = x^{-1/2}\big(3+4x+x^2\big) = \dfrac{(x+1)(x+3)}{\sqrt{x}}$.

3. Simplify: $\dfrac{y^{-1}-x^{-1}}{x^{-2}-y^{-2}} = \dfrac{\frac{x-y}{xy}}{\frac{y^2-x^2}{x^2y^2}} = \dfrac{x-y}{xy}\cdot\dfrac{x^2y^2}{(y-x)(y+x)} = \dfrac{-xy}{x+y}$.

6 Exponentials and Logarithms

Definition (logarithm)

$\log_b(x)$ is the exponent you put on b to get x: $\log_b(x)=y \iff b^y=x$ $(b>0,\ b\neq 1,\ x>0)$.
$\ln(x)=\log_e(x)$. Inverses: $b^{\log_b(x)}=x$ and $\log_b(b^x)=x$.

Key facts

- $\log_b(MN)=\log_b M+\log_b N$
- $\log_b\left(\frac{M}{N}\right)=\log_b M-\log_b N$
- $\log_b(M^p)=p\log_b M$
- $\log_b(1)=0,\ \log_b(b)=1$
- change of base: $\log_b(x)=\dfrac{\ln x}{\ln b}$
- there is *no* rule for $\log_b(M+N)$!

Examples

1. $\log_2(48)-\log_2(6)=\log_2\left(\frac{48}{6}\right)=\log_2(8)=3$.

2. Solve $e^{2x+3}=1$: take ln of both sides: $2x+3=0$, so $x=-\frac{3}{2}$.

3. Solve $2^x=3$: $x=\log_2(3)=\dfrac{\ln 3}{\ln 2}$ (exact — leave it this way, not 1.585).

4. Solve $\log_3(x)+\log_3(2x+1)=1$: combine: $\log_3\big(x(2x+1)\big)=1$, so $2x^2+x=3$, $(2x+3)(x-1)=0$, giving $x=1$ or $x=-\frac{3}{2}$. Check the domain: $x>0$, so only $\boxed{x=1}$.

Example (exponential growth — life sciences)

A bacteria culture starts at 1000 cells and doubles every 20 minutes: $N(t)=1000\cdot 2^{t/20}$. When does $N=5000$?

$$2^{t/20}=5 \;\Rightarrow\; \frac{t}{20}=\log_2(5) \;\Rightarrow\; t=20\,\frac{\ln 5}{\ln 2}\approx 46.4 \text{ minutes.}$$

Exponentials model growth/decay; logarithms answer "when?" — you will use this pair all semester.