Sections 5.1–5.3: Putting It Together — f, f', and f''

Synthesized examples: reading increase, decrease, extrema, concavity, and inflection points from formulas and from graphs

Strategy: everything about f from f' and f''

The same four steps, used twice:

	Part 1: the sign of f' (increasing/decreasing, local extrema)	**Part 2: the sign of f''** (concavity, inflection points)
1	find $f'(x)$, as one factored fraction	
2	critical points: $f' = 0$ or f' DNE	
3	sign chart for f': $\oplus$ increasing, $\ominus$ decreasing	
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Always

Only points *in the domain* of f can be cps or inflection points (VAs still split the number line).
Check: at a cp with $f'(c) = 0$, the second derivative test must agree with Part 1.

Example 1: everything about $f(x) = 2x + \frac{8}{x^2}$

Example

Given $f(x) = 2x + \dfrac{8}{x^2}$, $f'(x) = \dfrac{2\left(x^3 - 8\right)}{x^3}$, $f''(x) = \dfrac{48}{x^4}$, find

(a) the intervals where $f(x)$ is increasing and decreasing

(b) the x-coordinate of all local extrema

(c) the intervals where $f(x)$ is concave up and down

(d) the x-coordinate of all inflection points of $f(x)$

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Check the given derivatives: $f(x) = 2x + 8x^{-2}$, so
$f'(x) = 2 - 16x^{-3} = \frac{2x^3 - 16}{x^3} = \frac{2(x^3 - 8)}{x^3}$ and $f''(x) = 48x^{-4} = \frac{48}{x^4}$.

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Example 1, Part 1, Step 2: critical points of $f(x)$

$f'(x) = \dfrac{2\left(x^3 - 8\right)}{x^3}$. Critical points: interior points of the domain where $f'(x) = 0$ or $f'(x)$ DNE.

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Where $f'(x) = 0$ (numerator $= 0$):

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2 is in the domain of f: the **only** cp.

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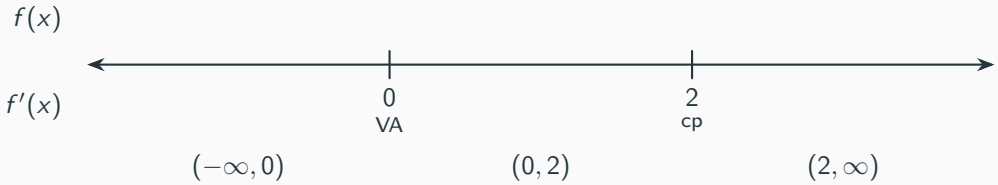
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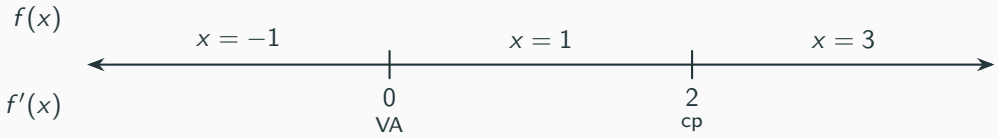
Number line for f'

Split at $x = 0$ (VA) and $x = 2$ (cp): intervals $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$.

Example 1, Part 1, Step 3: sign chart for $f'(x)$

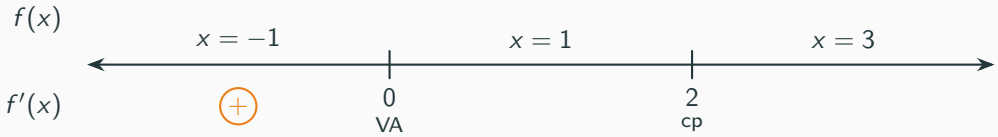


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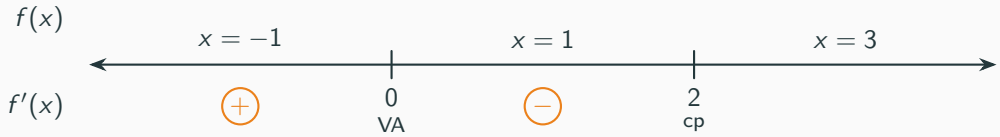
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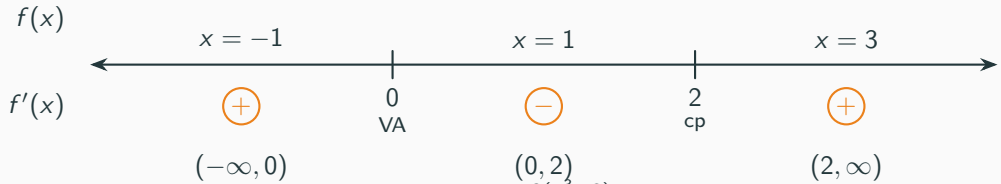


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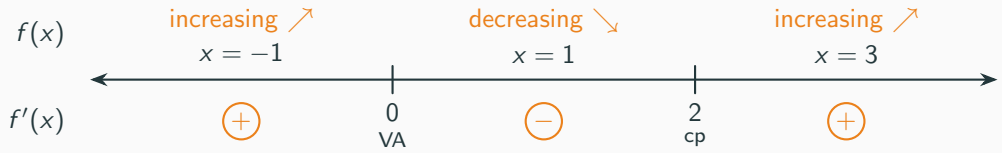
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Example 1, Part 1, Step 4: answers to (a) and (b)

(a) Increasing/decreasing

$f(x)$ is increasing on $(-\infty, 0)$ and $(2, \infty)$, and decreasing on $(0, 2)$.

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$x = 2$ gives a **local min**, since $f'(x)$ changes from $\ominus$ to $\oplus$ at $x = 2$. (The value is $f(2) = 4 + \frac{8}{4} = 6$.)

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- f' also changes sign at $x = 0$ (from $\oplus$ to $\ominus$), but $x = 0$ is not in the domain: no local max there.
- Second derivative test check: $f'(2) = 0$ and $f''(2) = \frac{48}{16} > 0$, so a local min — the same answer.

Example 1, Part 2, Step 2: candidates for inflection points

$f''(x) = \dfrac{48}{x^4}$. Find the cps of f': where $f''(x) = 0$ or $f''(x)$ DNE.

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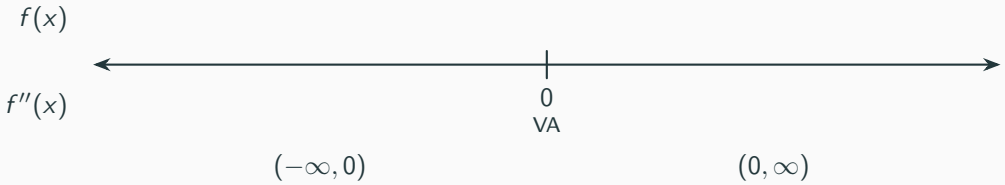
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Number line for f''

Split only at $x = 0$: intervals $(-\infty, 0)$ and $(0, \infty)$. No inflection point candidates at all.

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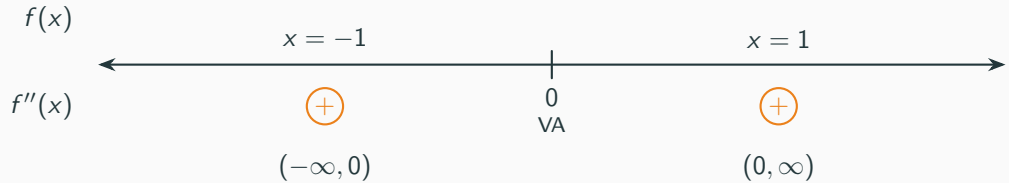
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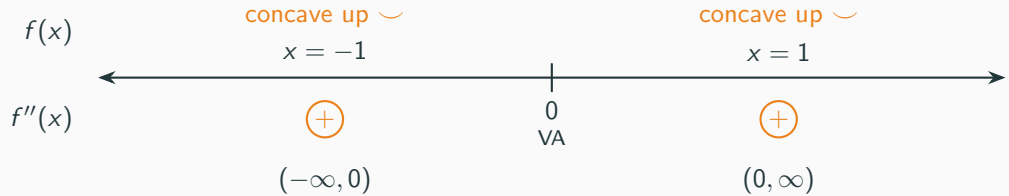


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(No surprise: x^4 is positive for every $x \neq 0$, so $f''(x) > 0$ everywhere it is defined.)

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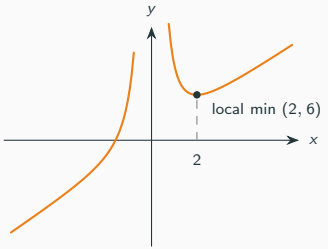
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The graph of f: both branches bend upward.

Strategy: reading f from the graph of f'

To find this about f ...	... look at this on the graph of f'
critical points	where it crosses or touches the x-axis ($f' = 0$)
increasing / decreasing	where it is above / below the x-axis
local max / local min	where it crosses from above to below / from below to above
concave up / concave down	where it is rising / falling
inflection points	its turning points (hilltops and valley bottoms)

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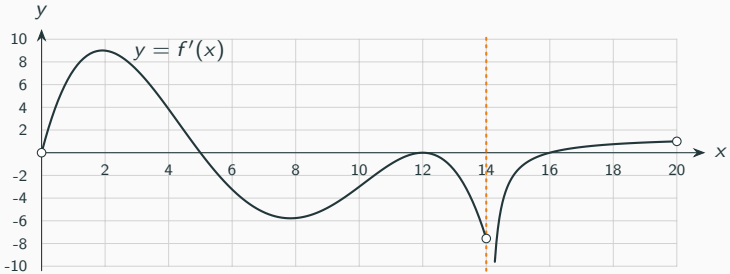
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Two things to read

The **height** of f' (above or below the axis) answers the first three rows; its **direction** (rising or falling) answers the last two. Endpoints and VAs are never cps or inflection points.

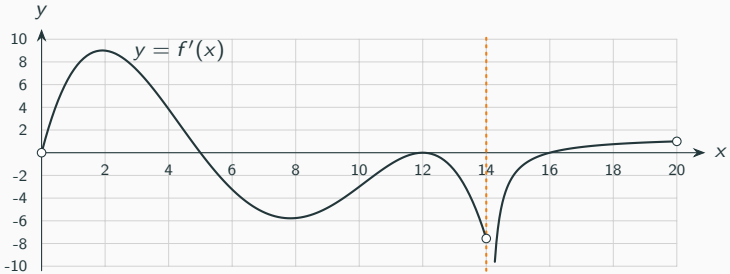
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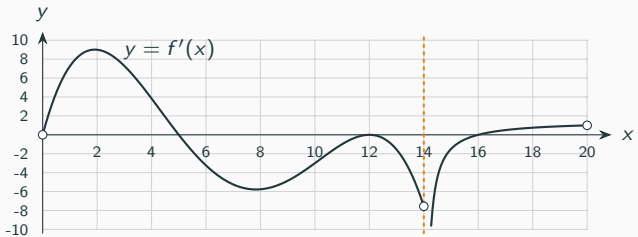
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Remember: this is the graph of the derivative, not of f. Every question about f is answered by reading heights and slopes of this graph.

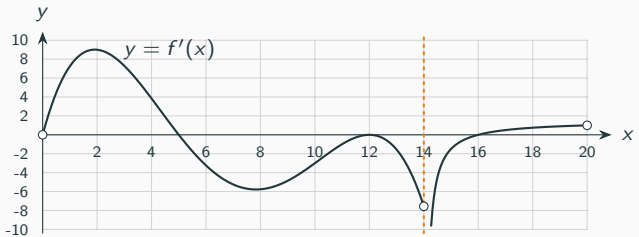
Example 2 (a): the critical points of $f(x)$



(a) Find the cps of $f(x)$

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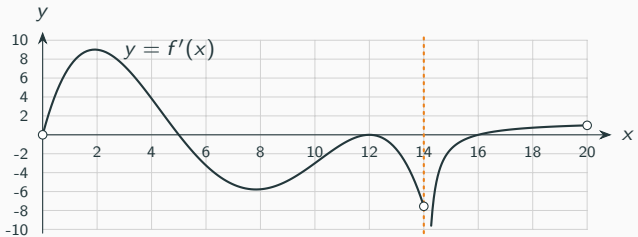


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$f'(x) = 0$ where the graph crosses or touches the x-axis: $x = 5,\ 12,\ 16$.

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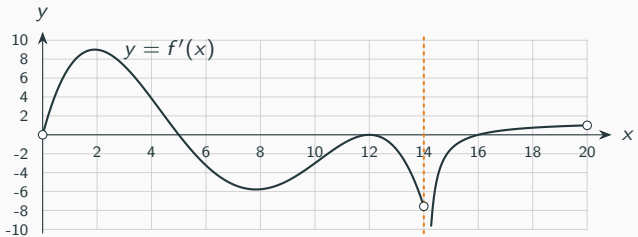
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Not cps: $x = 0$ and 20 are endpoints, and $x = 14$ is not in the domain of f.

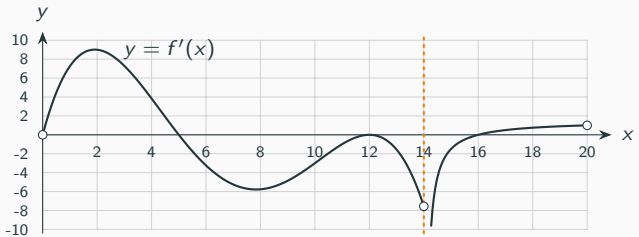
Example 2 (b): where $f(x)$ is increasing and decreasing



(b) Increasing/decreasing

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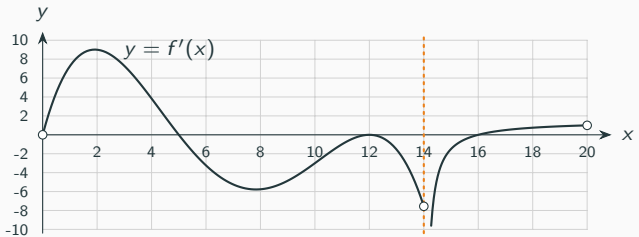
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Answer

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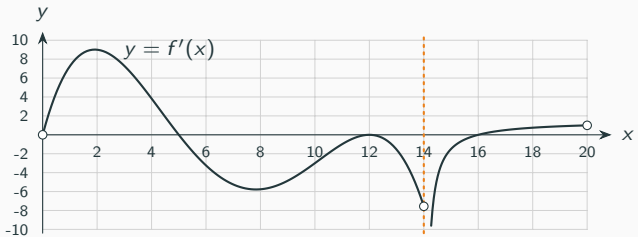
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Split at 14 (not in the domain); $f'(12) = 0$ does not split $(5, 14)$.

Example 2 (c): the local extrema of $f(x)$

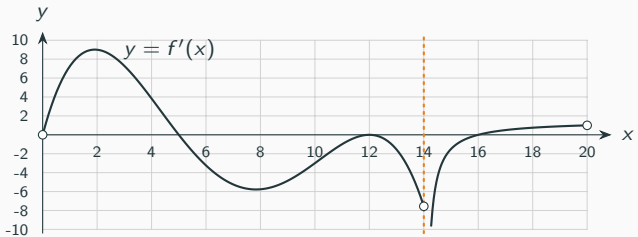


(c) Local extrema

First derivative test at each cp

$x = 5, 12, 16$:

Example 2 (c): the local extrema of $f(x)$

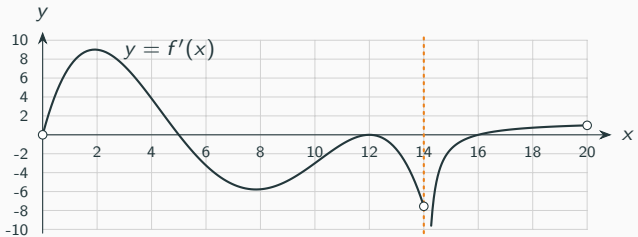


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- **local max at** $x = 5$: f' goes $\oplus$ to $\ominus$.

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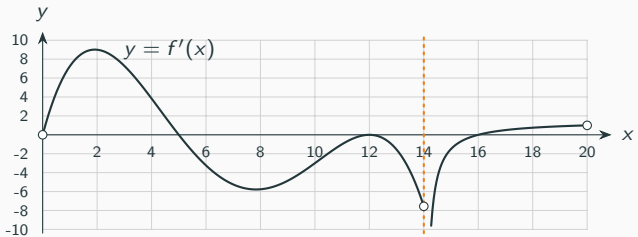


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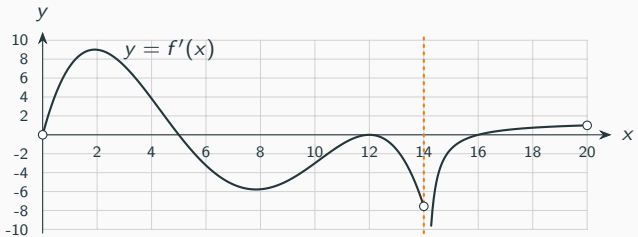


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- **local max at** $x = 5$: f' goes $\oplus$ to $\ominus$.
- **local min at** $x = 16$: f' goes $\ominus$ to $\oplus$.
- **none at** $x = 12$: f' touches 0, stays $\ominus$.

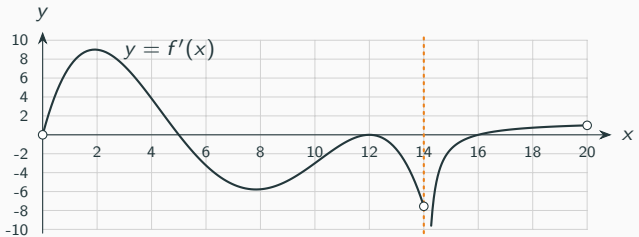
Example 2 (d): where $f(x)$ is concave up and concave down



(d) Concavity

Concave up where f' is **increasing** (graph of f' going up); concave down where f' is **decreasing**.

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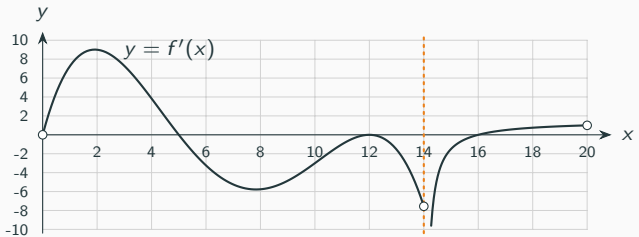
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Answer

concave up on $(0, 2)$, $(8, 12)$, $(14, 20)$;
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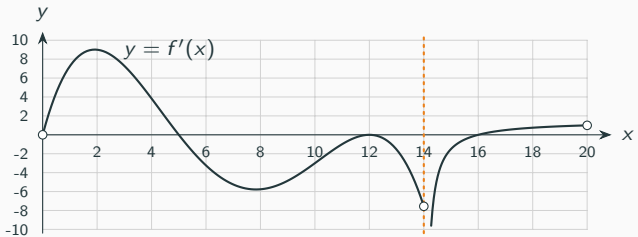
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Look at the *direction* of the f' graph here, not its sign.

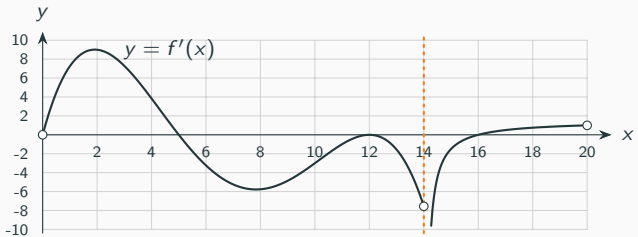
Example 2 (e): the inflection points of $f(x)$



(e) Inflection points

Where f' switches between increasing and decreasing: the **turning points** of the graph of f' (in the domain of f).

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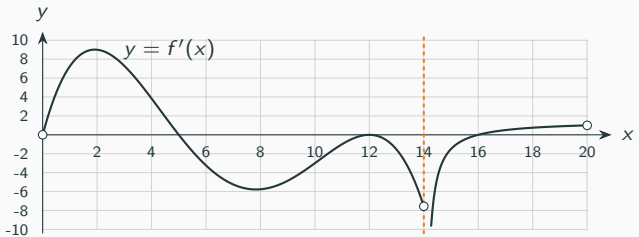
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Answer

$x = 2,\ 8,\ 12$

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(e) Inflection points

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Answer

$x = 2,\ 8,\ 12$

Not $x = 14$: the concavity changes across it, but 14 is not in the domain of f.

Strategy: which graph is f, f', and f''?

Strategy

1. Pick a curve and mark its **turning points** (the tops of hills and the bottoms of valleys).

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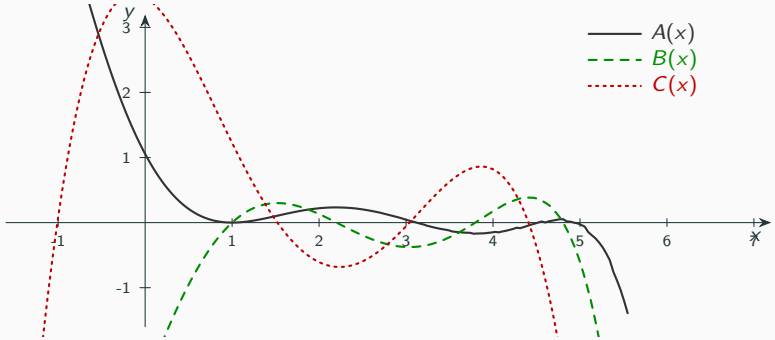
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Do not guess from appearances

Line style, color, and height say nothing about which curve is which: only zeros $\leftrightarrow$ turning points and signs $\leftrightarrow$ directions decide.

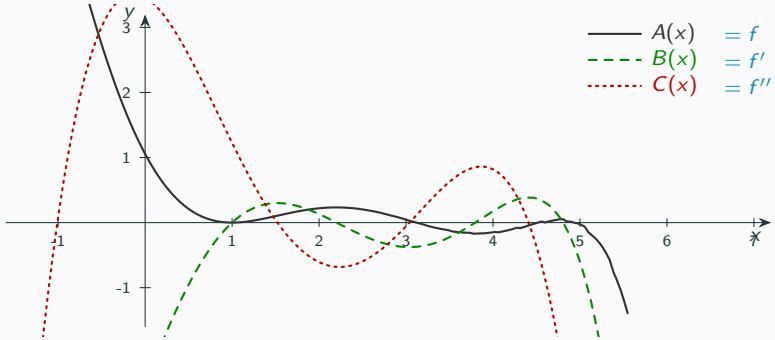
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The figure shows the graphs of f, f', and f''. Identify which graph is that of f''.



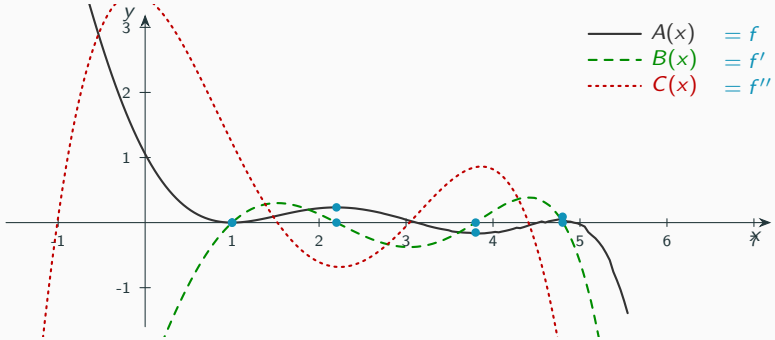
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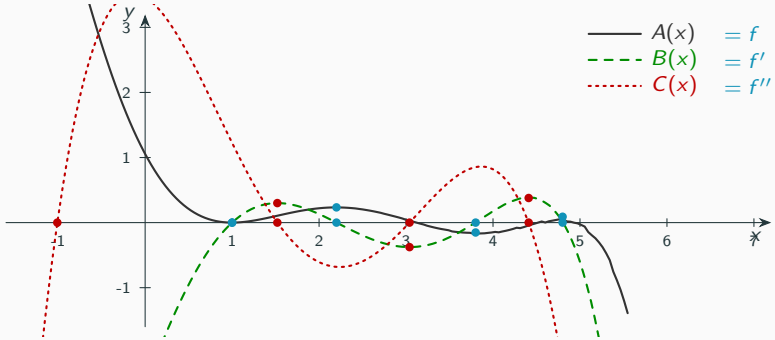
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Example 3 (a), continued: the reasoning

The two connections to look for

- **local extrema of a function** $\longleftrightarrow$ **zeros** of its derivative
- **intervals of increase/decrease** $\longleftrightarrow$ where the derivative is **positive/negative**

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1. **Steps 1–3:** A has turning points at $x \approx 1,\ 2.2,\ 3.8,\ 4.8$, and B is **zero** exactly there ($B < 0$ where A falls, $B > 0$ where A rises). So $B = A'$.

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2. **Step 4:** B has turning points at $x \approx 1.5,\ 3,\ 4.4$ (and one further left), and C is **zero** exactly there. So $C = B'$.

Example 3 (a), continued: the reasoning

The two connections to look for

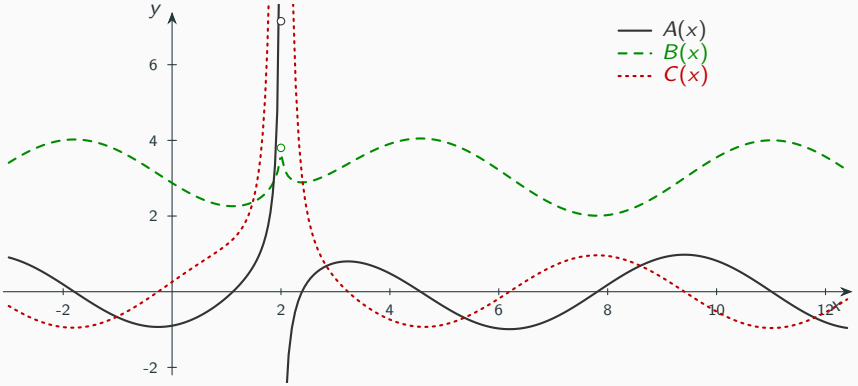
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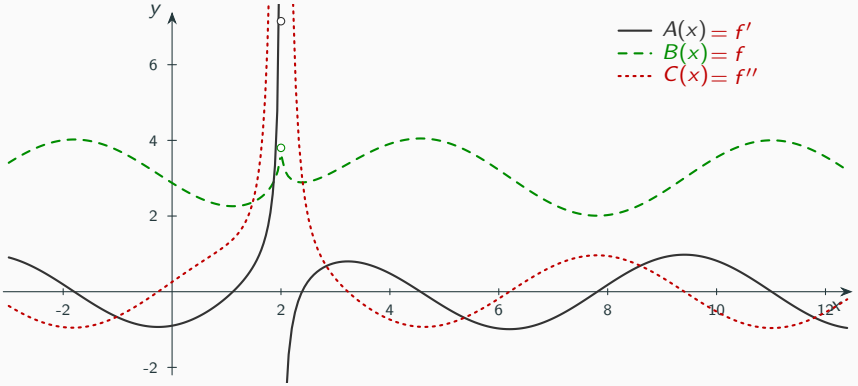
Conclusion

$A = f$, $B = f'$, $C = f''$: the graph of f'' is $C(x)$ (the dotted one). (Check: C is *not* zero at the turning points of A, so $C \neq A'$.)

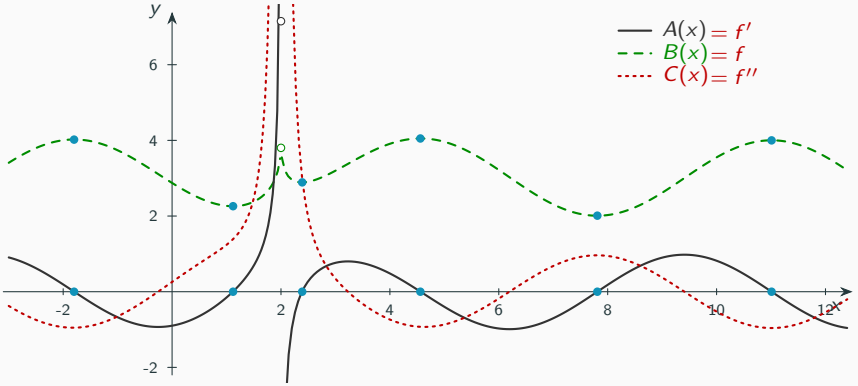
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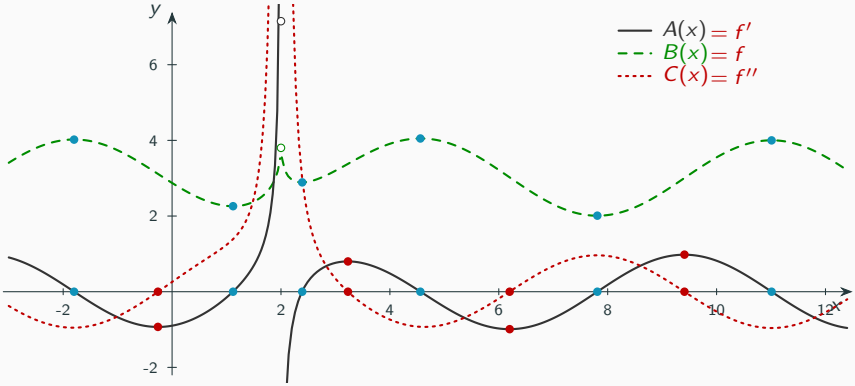
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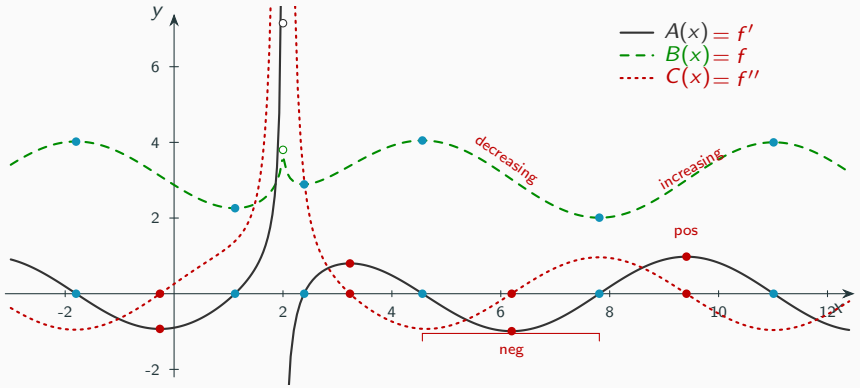
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Example 3 (b), continued: the reasoning

1. **Steps 1–3:** The dashed wave B has its hills and valleys exactly where the solid curve A crosses zero, and B is **decreasing** (from $x \approx 4.6$ to 7.8) exactly where A is **negative**, **increasing** (after 7.8) where A is **positive**. So $A = B'$.

Example 3 (b), continued: the reasoning

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2. **Step 4:** The solid curve A has its turning points exactly where the dotted curve C crosses zero. So $C = A'$.

Example 3 (b), continued: the reasoning

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2. **Step 4:** The solid curve A has its turning points exactly where the dotted curve C crosses zero. So $C = A'$.
3. The sharp peak of B at $x = 2$ fits: a corner in f makes f' blow up there (the solid curve shoots off to $\pm\infty$ on the two sides of $x = 2$) and f'' spike.

Example 3 (b), continued: the reasoning

1. **Steps 1–3:** The dashed wave B has its hills and valleys exactly where the solid curve A crosses zero, and B is **decreasing** (from $x \approx 4.6$ to 7.8) exactly where A is **negative**, **increasing** (after 7.8) where A is **positive**. So $A = B'$.
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3. The sharp peak of B at $x = 2$ fits: a corner in f makes f' blow up there (the solid curve shoots off to $\pm\infty$ on the two sides of $x = 2$) and f'' spike.

Conclusion

$B = f$, $A = f'$, $C = f''$: f'' is again $C(x)$ (dotted) — but this time the *solid* curve is f', not f. Moral: match zeros to turning points and signs to directions; don't assume the solid curve is f.

Summary: the dictionary between f, f', and f''

Feature of f	Read from f'	Read from f''
increasing / decreasing	$f' > 0$ / $f' < 0$	—
critical point	$f' = 0$ or f' DNE (in the domain)	—
local max / local min	sign of f': $\oplus$ to $\ominus$ / $\ominus$ to $\oplus$	$f' = 0$ with $f'' < 0$ / $f'' > 0$
concave up / down	f' increasing / decreasing	$f'' > 0$ / $f'' < 0$
inflection point	turning point of f'	sign change of f'' (in the domain)

- Points outside the domain (VAs) split the number line but are never cps or inflection points.
- Given the graph of f': its heights answer increase/decrease and extrema, its slopes answer concavity and inflection points.
- Identifying f, f', f'' from graphs: the zeros of each derivative sit under the turning points of the function before it.