Sections 5.1 & 5.2: Increasing/Decreasing and Relative Extrema

Using the sign of the derivative to read the shape of a graph

A motivating example: population growth

Example. Suppose the size of a population at time t is denoted

$$N(t) = \frac{100}{1 + 3e^{-2t}} \qquad \text{for } t \geq 0.$$

Question

Determine where $N(t)$ is increasing and where it is decreasing.

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This is the **logistic equation** from last time (its graph levels off at the carrying capacity 100). Today we build the tools to answer the question with the *derivative*, and we come back to $N(t)$ at the end.

Definition: increasing and decreasing

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Let f be a function defined on some interval. Then for any two numbers x_1 and x_2 in the interval,

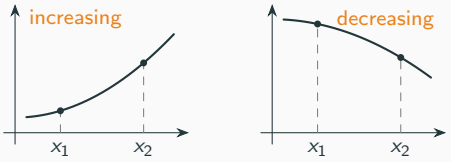
- f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, and
- f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

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Increasing: the graph goes **up** as you move to the right.

Decreasing: the graph goes **down** as you move to the right.

Theorem: the sign of $f'(x)$ tells the direction

Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then

1. If $f'(x) > 0$ for each x in (a, b), then $f(x)$ is increasing on (a, b).

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Why it makes sense

$f'(x)$ is the slope of the tangent line. Positive slopes mean the graph is climbing, negative slopes mean it is falling, and zero slope everywhere means a horizontal line.

Definition: local maximum and local minimum

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Suppose c is an **interior point** of some interval I for which $f(x)$ is defined.

- If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is called a local maximum.
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A local maximum is the top of a "hill" and a local minimum is the bottom of a "valley" (together: **local extrema**, also called **relative extrema**).

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Important

Local extrema **cannot** occur at the endpoints of an interval: c must have points of the interval on *both* sides of it.

Definition: critical points

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An interior point $x = c$ of the domain of $f(x)$ at which

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ DNE}$$

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Note — the converse is false

If c is a cp of $f(x)$, then $f(c)$ is **NOT** necessarily a local extremum! So the cps are the only *candidates* for local extrema; each candidate still has to be tested.

The first derivative test, case 1: no sign change

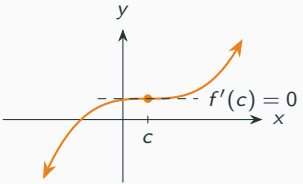
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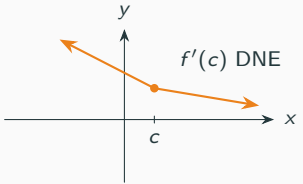
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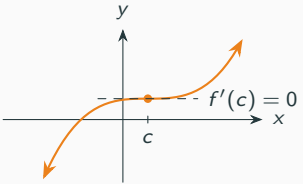


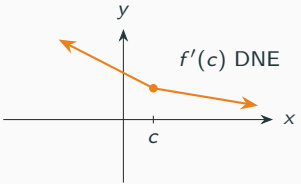


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On both graphs f is heading the same way on both sides of c, so c is neither a hill top nor a valley bottom.

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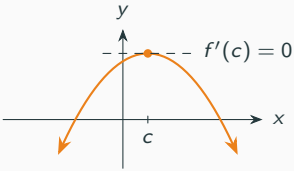
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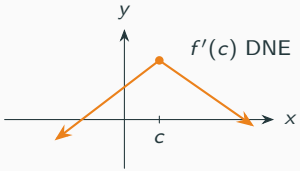
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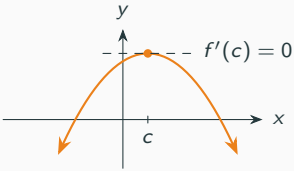


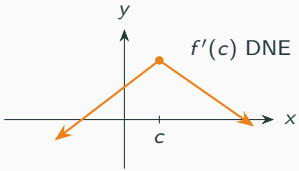


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Rising ($f' > 0$) before c, falling ($f' < 0$) after c: the top of a hill.

The first derivative test, case 3: $-$ to $+$

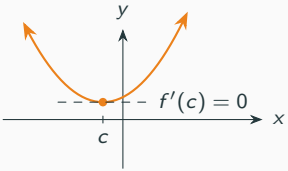
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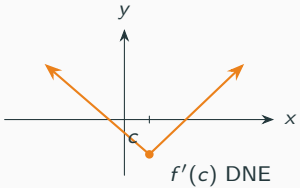
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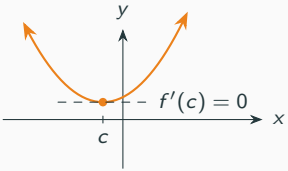


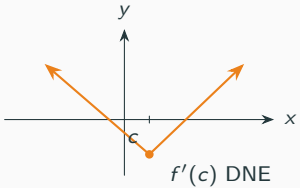


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Falling ($f' < 0$) before c, rising ($f' > 0$) after c: the bottom of a valley.

Strategy: increasing, decreasing, and local extrema

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Test-point shortcut: only the sign matters

Count the minus signs of the factors: e.g. $\frac{-x+16}{x^3}$ at $x = -1$ is $\frac{(+)}{(-)}$, so $\ominus$.

Example: $f(x) = \frac{x-8}{x^2}$ — Step 1: find $f'(x)$

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Example, Step 2: the critical points of $f(x)$

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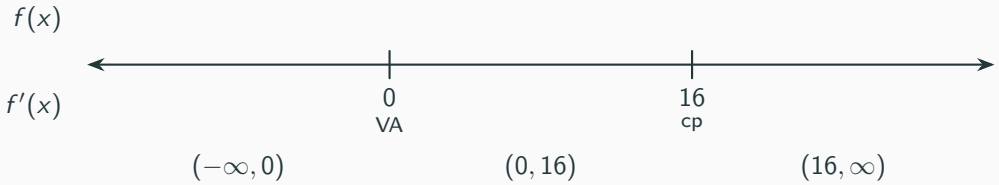
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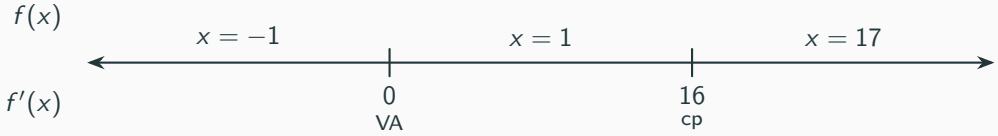
But $x = 0$ still goes on the number line

The sign of $f'(x)$ can change at a cp *or* at a break in the domain, so both $x = 0$ (VA) and $x = 16$ (cp) split the number line in the sign chart.

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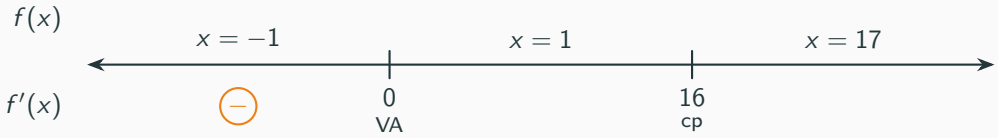


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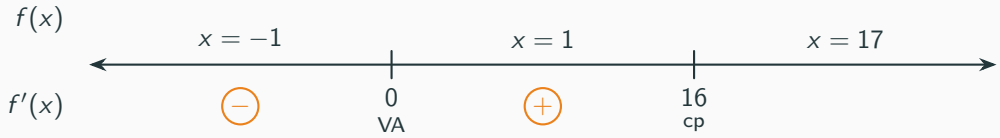
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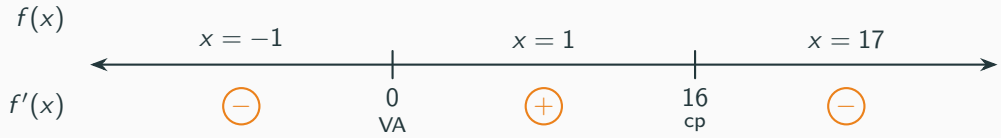


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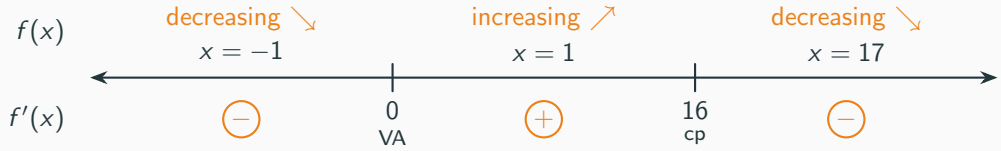
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Example, Step 4: read off the answer

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$f(x)$ has a **local max at** $x = 16$, since $f'(x)$ goes from $\oplus$ to $\ominus$ at $x = 16$.

- Nothing happens at $x = 0$: the sign of f' does change there, but $x = 0$ is not in the domain of f, so it is not a cp and $f(0)$ is not a local extremum (there is no $f(0)$!).
- Only the x-coordinate was asked for; the local maximum *value* is $f(16) = \frac{8}{256} = \frac{1}{32}$.

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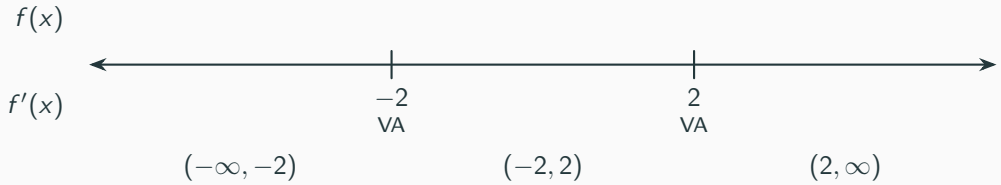
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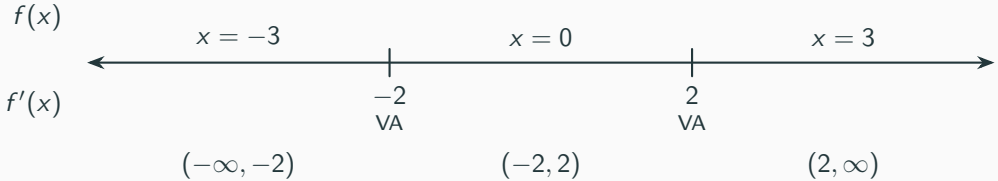
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So $f(x)$ has no critical points at all. The VAs at ± 2 still split the number line.

Try it, Step 3: sign chart for $f'(x)$

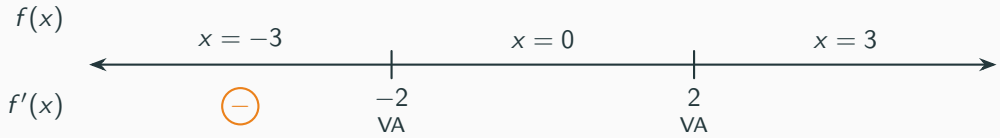


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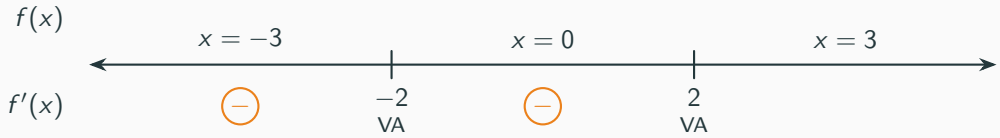
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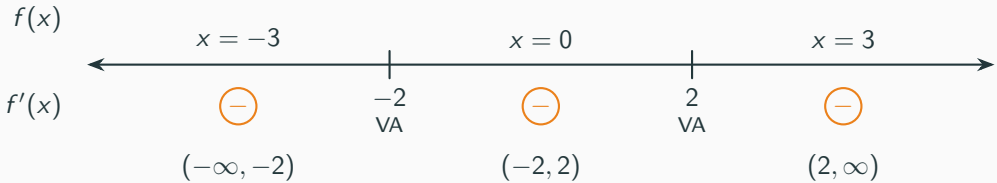


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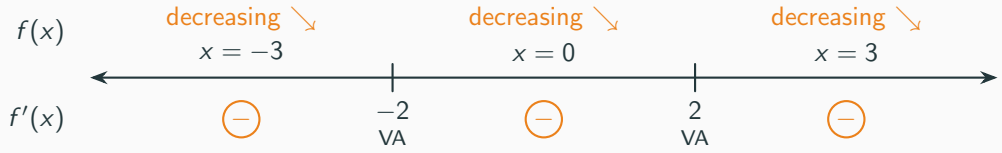
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Intervals of increase and decrease

$f(x)$ is decreasing on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

(No surprise: the numerator $-(x^2 + 4)$ is always negative and the denominator is a square, so $f'(x) < 0$ wherever it is defined.)

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Local extrema

Since there are **no cps**, there can be **no local extrema**.

Back to the population, Step 1: find $N'(t)$

Try it

$N(t) = \dfrac{100}{1 + 3e^{-2t}}$ for $t \geq 0$. Determine where $N(t)$ is increasing and decreasing.

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$$\begin{aligned} N'(t) &= \frac{0 \cdot \left(1 + 3e^{-2t}\right) - 100\left(-2 \cdot 3e^{-2t}\right)}{\left(1 + 3e^{-2t}\right)^2} \\ &= \frac{600e^{-2t}}{\left(1 + 3e^{-2t}\right)^2} \end{aligned}$$

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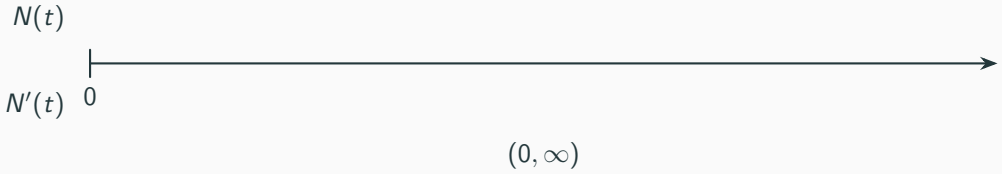
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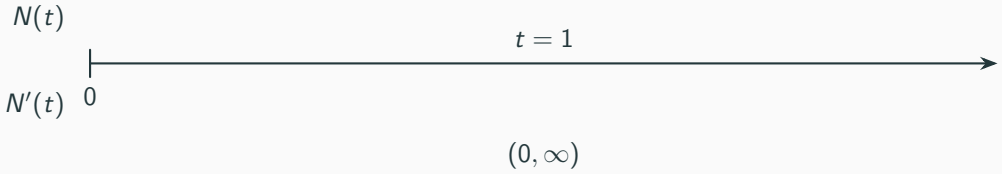
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So there are no cps, and nothing splits the number line: one interval, $(0, \infty)$.

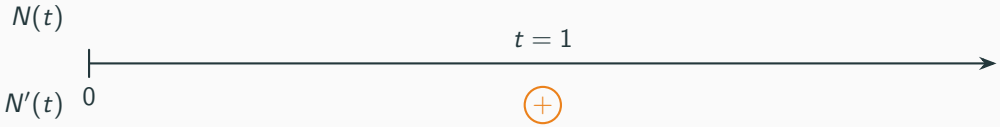
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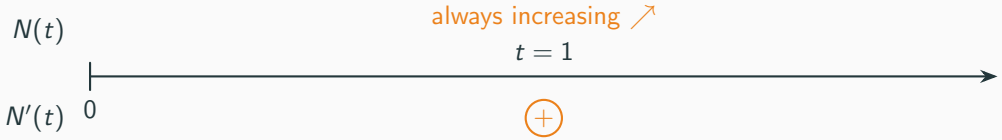


Population, Steps 3 and 4: sign chart and conclusion



$t = 1$: $\quad N'(1) = \dfrac{600e^{-2(1)}}{\left(1 + 3e^{-2(1)}\right)^2} \Rightarrow \oplus$ (positive over positive).

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Answer

$N(t)$ is **always increasing** on $(0, \infty)$, and there can be **no local extrema** since there are no cps. The population keeps growing toward the carrying capacity 100. We'll come back to this example next time!

Summary

The strategy for increasing/decreasing and local extrema

1. Find $f'(x)$.
2. Find the **critical points**: interior points of the domain where $f'(x) = 0$ or $f'(x)$ DNE (a zero of the denominator outside the domain is a VA, not a cp — but it still splits the number line).
3. Make a **sign chart**: test one point in each interval; $f' > 0$ means increasing, $f' < 0$ means decreasing.
4. Apply the **first derivative test** at each cp: $\oplus$ to $\ominus$ local max, $\ominus$ to $\oplus$ local min, no sign change no extremum.

- Local extrema can only occur at cps, but not every cp gives a local extremum.
- List intervals separately across vertical asymptotes (e.g. $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$).
- Logistic growth $N(t) = \frac{100}{1+3e^{-2t}}$: $N'(t) > 0$ always, so the population is always increasing.