Sections 4.1 & 4.2: The Derivative Rules

Basic derivatives, constant multiples, sums and differences, products, and quotients

The derivative rules: the basic functions

$f(x)$	$f'(x)$	example
constant (a number by itself)	0	$\frac{d}{dx}[-32] = 0$
x^n (**power rule**)	$n\,x^{n-1}$	$\frac{d}{dx}\left[x^{-5}\right] = -5x^{-5-1} = -5x^{-6}$
e^x	e^x	
$\ln(x)$	$\frac{1}{x}$	

The derivative rules: the basic functions

$f(x)$	$f'(x)$	example
constant (a number by itself)	0	$\frac{d}{dx}[-32] = 0$
x^n (**power rule**)	$n\,x^{n-1}$	$\frac{d}{dx}\left[x^{-5}\right] = -5x^{-5-1} = -5x^{-6}$
e^x	e^x	
$\ln(x)$	$\frac{1}{x}$	

Note

$\frac{d}{dx}\left[\,\cdots\right]$ means "the derivative *with respect to* x" of whatever is in the brackets.

The derivative rules: the trigonometric functions

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\csc(x)\cot(x)$

More “complicated” rules: (1) a constant multiplying a function

Rule 1: $c \cdot f(x)$ — a constant multiplying a function

$$\frac{d}{dx}[c\,f(x)] = c \cdot f'(x)$$

More “complicated” rules: (1) a constant multiplying a function

Rule 1: $c \cdot f(x)$ — a constant multiplying a function

$$\frac{d}{dx}[c\, f(x)] = c \cdot f'(x)$$

e.g.

$$\frac{d}{dx}\Big[\underbrace{3\ln(x)}_{c \cdot f(x)}\Big] = 3 \cdot \frac{1}{x} = \frac{3}{x}$$

More “complicated” rules: (2) a sum or difference of functions

Rule 2: $u(x) \pm v(x)$ — a sum/difference of functions

$$\frac{d}{dx}[u(x) \pm v(x)] = u'(x) \pm v'(x)$$

More “complicated” rules: (2) a sum or difference of functions

Rule 2: $u(x) \pm v(x)$ — a sum/difference of functions

$$\frac{d}{dx}[u(x) \pm v(x)] = u'(x) \pm v'(x)$$

e.g.

$$\frac{d}{dx}\Big[\underbrace{x^3 + e^x}_{u(x)+v(x)}\Big] = \frac{d}{dx}[x^3] + \frac{d}{dx}[e^x]$$

More “complicated” rules: (2) a sum or difference of functions

Rule 2: $u(x) \pm v(x)$ — a sum/difference of functions

$$\frac{d}{dx}[u(x) \pm v(x)] = u'(x) \pm v'(x)$$

e.g.

$$\frac{d}{dx}\Big[\underbrace{x^3 + e^x}_{u(x)+v(x)}\Big] = \frac{d}{dx}[x^3] + \frac{d}{dx}[e^x]$$

$$= 3x^2 + e^x$$

More “complicated” rules: (3) a product of two functions

Rule 3: $u(x) \cdot v(x)$ — a product of two functions

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x)\, v(x) + u(x)\, v'(x)$$

More “complicated” rules: (3) a product of two functions

Rule 3: $u(x) \cdot v(x)$ — a product of two functions

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x)\, v(x) + u(x)\, v'(x)$$

e.g.

$$\frac{d}{dx}\Big[\underbrace{\sin(x) \cdot x^{-1}}_{u(x) \cdot v(x)}\Big] = \frac{d}{dx}[\sin(x)]\; x^{-1} + \sin(x)\; \frac{d}{dx}\left[x^{-1}\right]$$

More "complicated" rules: (3) a product of two functions

Rule 3: $u(x) \cdot v(x)$ — a product of two functions

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x)\, v(x) + u(x)\, v'(x)$$

e.g.

$$\frac{d}{dx}\Big[\underbrace{\sin(x) \cdot x^{-1}}_{u(x) \cdot v(x)}\Big] = \frac{d}{dx}[\sin(x)]\; x^{-1} + \sin(x)\; \frac{d}{dx}\left[x^{-1}\right]$$

$$= \cos(x) \cdot x^{-1} + \sin(x)\left(-x^{-2}\right)$$

More “complicated” rules: (4) a quotient of functions

Rule 4: $\dfrac{u(x)}{v(x)}$ — a quotient of functions

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{\left[v(x)\right]^2}$$

More “complicated” rules: (4) a quotient of functions

Rule 4: $\dfrac{u(x)}{v(x)}$ — a quotient of functions

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{\big[v(x)\big]^2}$$

e.g.

$$\frac{d}{dx}\left[\frac{\tan(x)}{\ln(x)}\right] = \frac{\dfrac{d}{dx}[\tan(x)]\,\ln(x) - \tan(x)\,\dfrac{d}{dx}[\ln(x)]}{\big[\ln(x)\big]^2}$$

More "complicated" rules: (4) a quotient of functions

Rule 4: $\dfrac{u(x)}{v(x)}$ — a quotient of functions

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{\big[v(x)\big]^2}$$

e.g.

$$\frac{d}{dx}\left[\frac{\tan(x)}{\ln(x)}\right] = \frac{\dfrac{d}{dx}[\tan(x)]\,\ln(x) - \tan(x)\,\dfrac{d}{dx}[\ln(x)]}{\big[\ln(x)\big]^2}$$

$$= \frac{\sec^2(x)\,\ln(x) - \tan(x)\cdot\dfrac{1}{x}}{\big(\ln(x)\big)^2}$$

Calculate $f'(x)$: **(a)** $f(x) = 10x^{12} - \frac{5}{x^3} + \sqrt{x}$

Try it: calculate $f'(x)$ for $f(x) = 10x^{12} - \dfrac{5}{x^3} + \sqrt{x}$.

Calculate $f'(x)$: **(a)** $f(x) = 10x^{12} - \frac{5}{x^3} + \sqrt{x}$

Try it: calculate $f'(x)$ for $f(x) = 10x^{12} - \dfrac{5}{x^3} + \sqrt{x}$.

Sum/Diff

$$f'(x) = \frac{d}{dx}\left[10\,x^{12}\right] - \frac{d}{dx}\left[\frac{5}{x^3}\right] + \frac{d}{dx}\left[\sqrt{x}\right]$$

Calculate $f'(x)$: **(a)** $f(x) = 10x^{12} - \frac{5}{x^3} + \sqrt{x}$

Try it: calculate $f'(x)$ for $f(x) = 10x^{12} - \dfrac{5}{x^3} + \sqrt{x}$.

Sum/Diff

$$f'(x) = \frac{d}{dx}\left[10\,x^{12}\right] - \frac{d}{dx}\left[\frac{5}{x^3}\right] + \frac{d}{dx}\left[\sqrt{x}\right]$$

we need to rewrite the last two as x^n

$$= 10\underbrace{\frac{d}{dx}\left[x^{12}\right]}_{x^n} - 5\underbrace{\frac{d}{dx}\left[x^{-3}\right]}_{x^n} + \underbrace{\frac{d}{dx}\left[x^{1/2}\right]}_{x^n}$$

Calculate $f'(x)$: **(a)** $f(x) = 10x^{12} - \frac{5}{x^3} + \sqrt{x}$

Try it: calculate $f'(x)$ for $f(x) = 10x^{12} - \dfrac{5}{x^3} + \sqrt{x}$.

Sum/Diff

$$f'(x) = \frac{d}{dx}\left[10\,x^{12}\right] - \frac{d}{dx}\left[\frac{5}{x^3}\right] + \frac{d}{dx}\left[\sqrt{x}\right]$$

we need to rewrite the last two as x^n

$$= 10\,\underbrace{\frac{d}{dx}\left[x^{12}\right]}_{x^n} - 5\,\underbrace{\frac{d}{dx}\left[x^{-3}\right]}_{x^n} + \underbrace{\frac{d}{dx}\left[x^{1/2}\right]}_{x^n}$$

$$= 10 \cdot 12\,x^{12-1} - 5(-3)\,x^{-3-1} + \tfrac{1}{2}\,x^{\frac{1}{2}-1}$$

Calculate $f'(x)$: **(a)** $f(x) = 10x^{12} - \frac{5}{x^3} + \sqrt{x}$

Try it: calculate $f'(x)$ for $f(x) = 10x^{12} - \dfrac{5}{x^3} + \sqrt{x}$. **Sum/Diff**

$$f'(x) = \frac{d}{dx}\left[10\,x^{12}\right] - \frac{d}{dx}\left[\frac{5}{x^3}\right] + \frac{d}{dx}\left[\sqrt{x}\right]$$

we need to rewrite the last two as x^n

$$= 10\,\underbrace{\frac{d}{dx}\left[x^{12}\right]}_{x^n} - 5\,\underbrace{\frac{d}{dx}\left[x^{-3}\right]}_{x^n} + \underbrace{\frac{d}{dx}\left[x^{1/2}\right]}_{x^n}$$

$$= 10 \cdot 12\,x^{12-1} - 5(-3)\,x^{-3-1} + \tfrac{1}{2}\,x^{\frac{1}{2}-1}$$

$$= 120x^{11} + 15x^{-4} + \tfrac{1}{2}\,x^{-1/2}$$

Caution!

CAUTION

$$f'(x) = 120x^{11} - 5x^{-3} + x^{1/2}$$

This is **half** $f'(x)$**, half** $f(x)$: the first term was differentiated, but the last two terms were just copied from $f(x)$.

Caution!

CAUTION

$$f'(x) = 120x^{11} - 5x^{-3} + x^{1/2}$$

This is **half** $f'(x)$**, half** $f(x)$: the first term was differentiated, but the last two terms were just copied from $f(x)$.

The correct answer differentiates *every* term:

$$f'(x) = 120x^{11} + 15x^{-4} + \tfrac{1}{2}\,x^{-1/2}.$$

Calculate $f'(x)$: **(b)** $f(x) = 3x^{13} + \frac{7}{\sqrt{x}} + \frac{5}{x^2} - e^{12}$

Try it: calculate $f'(x)$ for $f(x) = 3x^{13} + \dfrac{7}{\sqrt{x}} + \dfrac{5}{x^2} - e^{12}$.

Calculate $f'(x)$: **(b)** $f(x) = 3x^{13} + \frac{7}{\sqrt{x}} + \frac{5}{x^2} - e^{12}$

Try it: calculate $f'(x)$ for $f(x) = 3x^{13} + \dfrac{7}{\sqrt{x}} + \dfrac{5}{x^2} - e^{12}$.

$$f'(x) = 3\underbrace{\frac{d}{dx}\left[x^{13}\right]}_{x^n} + 7\underbrace{\frac{d}{dx}\left[x^{-1/2}\right]}_{x^n} + 5\underbrace{\frac{d}{dx}\left[x^{-2}\right]}_{x^n} - \underbrace{\frac{d}{dx}\left[e^{12}\right]}_{c}$$

Calculate $f'(x)$: **(b)** $f(x) = 3x^{13} + \frac{7}{\sqrt{x}} + \frac{5}{x^2} - e^{12}$

Try it: calculate $f'(x)$ for $f(x) = 3x^{13} + \dfrac{7}{\sqrt{x}} + \dfrac{5}{x^2} - e^{12}$.

$$f'(x) = 3\underbrace{\frac{d}{dx}\left[x^{13}\right]}_{x^n} + 7\underbrace{\frac{d}{dx}\left[x^{-1/2}\right]}_{x^n} + 5\underbrace{\frac{d}{dx}\left[x^{-2}\right]}_{x^n} - \underbrace{\frac{d}{dx}\left[e^{12}\right]}_{c}$$

$$= 3 \cdot 13\, x^{12} + 7\left(-\tfrac{1}{2}\right) x^{-3/2} + 5(-2)\, x^{-3} - 0$$

Calculate $f'(x)$: **(b)** $f(x) = 3x^{13} + \frac{7}{\sqrt{x}} + \frac{5}{x^2} - e^{12}$

Try it: calculate $f'(x)$ for $f(x) = 3x^{13} + \dfrac{7}{\sqrt{x}} + \dfrac{5}{x^2} - e^{12}$.

$$
\begin{aligned}
f'(x) &= 3\underbrace{\frac{d}{dx}\left[x^{13}\right]}_{x^n} + 7\underbrace{\frac{d}{dx}\left[x^{-1/2}\right]}_{x^n} + 5\underbrace{\frac{d}{dx}\left[x^{-2}\right]}_{x^n} - \underbrace{\frac{d}{dx}\left[e^{12}\right]}_{c} \\
&= 3 \cdot 13\, x^{12} + 7\left(-\tfrac{1}{2}\right) x^{-3/2} + 5(-2)\, x^{-3} - 0 \\
&= 39x^{12} - \tfrac{7}{2}\, x^{-3/2} - 10x^{-3}
\end{aligned}
$$

Calculate $f'(x)$: **(c)** $f(x) = 3x^7 \cos(x)$

Try it: calculate $f'(x)$ for $f(x) = 3x^7 \cos(x)$.

Calculate $f'(x)$: (c) $f(x) = 3x^7 \cos(x)$

Try it: calculate $f'(x)$ for $f(x) = 3x^7 \cos(x)$. **Product**

"Count your x's": $3x^7$ and $\cos(x)$ are two factors that each contain x, so this is a product.

$$f'(x) = \frac{d}{dx}\left[3x^7\right]\cos(x) + 3x^7 \frac{d}{dx}[\cos(x)]$$

Calculate $f'(x)$: (c) $f(x) = 3x^7 \cos(x)$

Try it: calculate $f'(x)$ for $f(x) = 3x^7 \cos(x)$. **Product**

"Count your x's": $3x^7$ and $\cos(x)$ are two factors that each contain x, so this is a product.

$$f'(x) = \frac{d}{dx}\left[3x^7\right]\cos(x) + 3x^7\,\frac{d}{dx}[\cos(x)]$$

The pieces

$$\frac{d}{dx}\left[3x^7\right] = 3\,\frac{d}{dx}\left[x^7\right] = 3 \cdot 7x^6 = 21x^6 \qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Calculate $f'(x)$: (c) $f(x) = 3x^7 \cos(x)$

Try it: calculate $f'(x)$ for $f(x) = 3x^7 \cos(x)$. **Product**

"Count your x's": $3x^7$ and $\cos(x)$ are two factors that each contain x, so this is a product.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[3x^7\right]\cos(x) + 3x^7\,\frac{d}{dx}[\cos(x)] \\ &= 21x^6\cos(x) + 3x^7\big(-\sin(x)\big) \end{aligned}$$

The pieces

$$\frac{d}{dx}\left[3x^7\right] = 3\,\frac{d}{dx}\left[x^7\right] = 3\cdot 7x^6 = 21x^6 \qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Calculate $f'(x)$: **(d)** $f(x) = 7\tan(x) \cdot x^{-4}$

Try it: calculate $f'(x)$ for $f(x) = 7\tan(x) \cdot x^{-4}$.

Calculate $f'(x)$: **(d)** $f(x) = 7\tan(x) \cdot x^{-4}$

Try it: calculate $f'(x)$ for $f(x) = 7\tan(x) \cdot x^{-4}$.

Product

$$f'(x) = \frac{d}{dx}[7\tan(x)]\ x^{-4} + 7\tan(x)\ \frac{d}{dx}\left[x^{-4}\right]$$

Calculate $f'(x)$: **(d)** $f(x) = 7\tan(x) \cdot x^{-4}$

Try it: calculate $f'(x)$ for $f(x) = 7\tan(x) \cdot x^{-4}$.

Product

$$\begin{aligned} f'(x) &= \frac{d}{dx}[7\tan(x)]\ x^{-4} + 7\tan(x)\ \frac{d}{dx}\left[x^{-4}\right] \\ &= 7\sec^2(x)\, x^{-4} + 7\tan(x)\left(-4x^{-5}\right) \end{aligned}$$

Calculate $f'(x)$: **(e)** $f(x) = \frac{8e^x}{x-3}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{8e^x}{x-3}$.

Calculate $f'(x)$: (e) $f(x) = \frac{8e^x}{x-3}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{8e^x}{x-3}$.

Quotient Rule

$$f'(x) = \frac{\frac{d}{dx}[8e^x]\,(x-3) - 8e^x\,\frac{d}{dx}[x-3]}{(x-3)^2}$$

Calculate $f'(x)$: (e) $f(x) = \frac{8e^x}{x-3}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{8e^x}{x-3}$.

Quotient Rule

$$f'(x) = \frac{\frac{d}{dx}[8e^x]\,(x-3) - 8e^x\,\frac{d}{dx}[x-3]}{(x-3)^2}$$

The pieces

$$\frac{d}{dx}[8e^x] = 8\,\frac{d}{dx}[e^x] = 8e^x \qquad \frac{d}{dx}[x-3] = \frac{d}{dx}\left[x^1\right] - \frac{d}{dx}[3] = 1x^{1-1} - 0 = x^0 = 1$$

Calculate $f'(x)$: (e) $f(x) = \frac{8e^x}{x-3}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{8e^x}{x-3}$.

Quotient Rule

$$f'(x) = \frac{\frac{d}{dx}[8e^x]\,(x-3) - 8e^x\,\frac{d}{dx}[x-3]}{(x-3)^2}$$

$$= \frac{8e^x(x-3) - 8e^x(1)}{(x-3)^2}$$

The pieces

$$\frac{d}{dx}[8e^x] = 8\,\frac{d}{dx}[e^x] = 8e^x \qquad \frac{d}{dx}[x-3] = \frac{d}{dx}\left[x^1\right] - \frac{d}{dx}[3] = 1x^{1-1} - 0 = x^0 = 1$$

Calculate $f'(x)$: **(f)** $f(x) = \frac{x^2 e^x}{\ln(x) - \tan(x)}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{x^2 e^x}{\ln(x) - \tan(x)}$.

Calculate $f'(x)$: **(f)** $f(x) = \frac{x^2 e^x}{\ln(x) - \tan(x)}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{x^2 e^x}{\ln(x) - \tan(x)}$.

Quotient

$$f'(x) = \frac{\frac{d}{dx}\left[x^2 e^x\right]\ \left(\ln(x) - \tan(x)\right) - \left(x^2 e^x\right)\frac{d}{dx}[\ln(x) - \tan(x)]}{\left(\ln(x) - \tan(x)\right)^2}$$

Calculate $f'(x)$: **(f)** $f(x) = \frac{x^2 e^x}{\ln(x) - \tan(x)}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{x^2 e^x}{\ln(x) - \tan(x)}$.

Quotient

$$f'(x) = \frac{\frac{d}{dx}\left[x^2 e^x\right]\ \left(\ln(x) - \tan(x)\right) - \left(x^2 e^x\right)\frac{d}{dx}[\ln(x) - \tan(x)]}{\left(\ln(x) - \tan(x)\right)^2}$$

The pieces

$$\frac{d}{dx}\left[x^2 e^x\right] = \frac{d}{dx}\left[x^2\right]\ e^x + x^2\,\frac{d}{dx}[e^x] = \left(2xe^x + x^2 e^x\right) \qquad \text{(a product inside!)}$$

$$\frac{d}{dx}[\ln(x) - \tan(x)] = \left(\frac{1}{x} - \sec^2(x)\right)$$

Calculate $f'(x)$: **(f)** $f(x) = \frac{x^2 e^x}{\ln(x) - \tan(x)}$, continued

$$f'(x) = \frac{\frac{d}{dx}\left[x^2 e^x\right]\ \left(\ln(x) - \tan(x)\right) - \left(x^2 e^x\right)\ \frac{d}{dx}[\ln(x) - \tan(x)]}{\left(\ln(x) - \tan(x)\right)^2}$$

Calculate $f'(x)$: **(f)** $f(x) = \frac{x^2 e^x}{\ln(x) - \tan(x)}$, continued

$$f'(x) = \frac{\frac{d}{dx}\left[x^2 e^x\right]\left(\ln(x) - \tan(x)\right) - \left(x^2 e^x\right)\frac{d}{dx}\left[\ln(x) - \tan(x)\right]}{\left(\ln(x) - \tan(x)\right)^2}$$

$$= \frac{\left(2xe^x + x^2 e^x\right)\left(\ln(x) - \tan(x)\right) - \left(x^2 e^x\right)\left(\frac{1}{x} - \sec^2(x)\right)}{\left(\ln(x) - \tan(x)\right)^2}$$

Summary

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
constant	0	$\sin(x)$	$\cos(x)$
x^n	$n\,x^{n-1}$	$\cos(x)$	$-\sin(x)$
e^x	e^x	$\tan(x)$	$\sec^2(x)$
$\ln(x)$	$1/x$	$\sec(x)$	$\sec(x)\tan(x)$
		$\cot(x)$	$-\csc^2(x)$
		$\csc(x)$	$-\csc(x)\cot(x)$

Structure	Rule
$c\,f(x)$	$c\,f'(x)$
$u \pm v$	$u' \pm v'$
$u \cdot v$	$u'v + uv'$
$\dfrac{u}{v}$	$\dfrac{u'v - uv'}{v^2}$

Big idea: first decide the structure of f (constant multiple? sum? product? quotient? "count your x's"), rewrite roots and reciprocals as x^n, then apply the matching rule to *every* term — not half $f'(x)$, half $f(x)$.