Sections 3.4 & 6.5: Linear Approximation

Using the tangent line to approximate a function near a point

A motivating example: drug concentration

The concentration of a certain drug in the bloodstream t hours after being administered is approximately

$$C(t) = \frac{5t}{9 + t^2}.$$

Try it

Approximate the change in concentration for $t = 1$ to $t = 1.5$.

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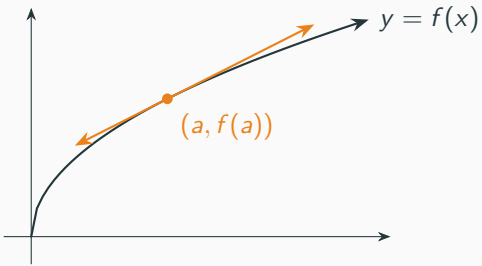
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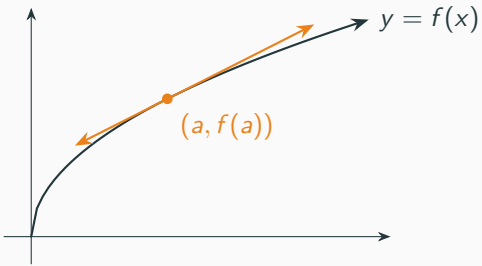
We want to approximate $C(1.5) - C(1)$ — and the word approximate is a hint that we do not have to compute it exactly. First we need one new idea; then we'll come back to this problem.

The idea of linear approximation



NEAR the point of tangency, the graph $y = f(x)$ and the tangent line through $(a, f(a))$ are **indistinguishable**.

The idea of linear approximation



NEAR the point of tangency, the graph $y = f(x)$ and the tangent line through $(a, f(a))$ are **indistinguishable**.

So we can use the **tangent line equation** to approximate the value of the function at points NEAR $(a, f(a))$.

Building the tangent line

To write the tangent line at $x = a$ we need two things:

1. **Slope:** $f'(a)$
2. **Point of tangency:** $(a, f(a))$

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$$y - f(a) = f'(a)(x - a) \qquad \text{call this } L(x)$$

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Definition

$L(x) = f'(a)(x - a) + f(a)$ is called a linear approximation for $f(x)$ at $x = a$:

$$f(x) \approx L(x) \quad \text{for } x\text{-values } \textbf{NEAR } a.$$

Example: approximate $\sqrt{109}$ using a linear approximation

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Set-up

$f(x) = \sqrt{x}$ and $a = 100$, since 109 is near 100 *and* we know $\sqrt{100} = 10$.

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1. **Slope:** $f'(x) = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$, so $f'(100) = \dfrac{1}{2\sqrt{100}} = \dfrac{1}{20}$.

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1. **Slope:** $f'(x) = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$, so $f'(100) = \dfrac{1}{2\sqrt{100}} = \dfrac{1}{20}$.
2. **Point of tangency:** $(100, f(100)) = (100, 10)$.

$\sqrt{109}$, continued: the tangent line

$$y - y_1 = m(x - x_1)$$

$\sqrt{109}$, continued: the tangent line

$$y - y_1 = m(x - x_1)$$
$$y - 10 = \frac{1}{20}(x - 100)$$

$\sqrt{109}$, continued: the tangent line

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 10 &= \tfrac{1}{20}(x - 100) \\
y &= \tfrac{1}{20}(x - 100) + 10 \qquad \text{this is } L(x)
\end{aligned}
$$

$\sqrt{109}$, continued: the tangent line

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 10 &= \tfrac{1}{20}(x - 100) \\
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\end{aligned}
$$

The linear approximation

$$L(x) = \frac{1}{20}(x - 100) + 10$$

$\sqrt{109}$, continued: use $L(x)$ in place of $f(x)$

Since 109 is near 100, $f(109) = \sqrt{109} \approx L(109)$.

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$$\sqrt{109} \approx L(109) = \tfrac{1}{20}(109 - 100) + 10 = \tfrac{1}{20}(9) + 10 = \tfrac{9}{20} + 10$$

$\sqrt{109}$, continued: use $L(x)$ in place of $f(x)$

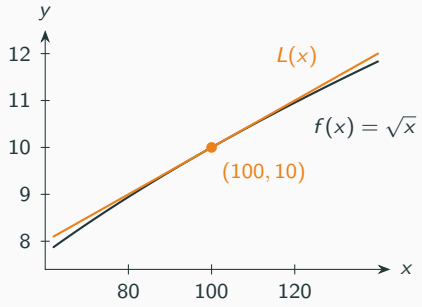
Since 109 is near 100, $\quad f(109) = \sqrt{109} \approx L(109)$.

$$\sqrt{109} \approx L(109) = \tfrac{1}{20}(109 - 100) + 10 = \tfrac{1}{20}(9) + 10 = \tfrac{9}{20} + 10$$

Answer

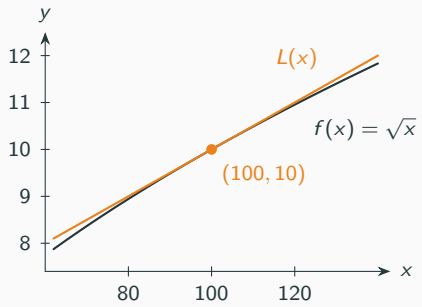
$\sqrt{109} \approx \frac{9}{20} + 10 = 10.45$ (a calculator gives $\sqrt{109} \approx 10.4403$)

Check it: graph $f(x) = \sqrt{x}$ with the tangent line



Graph $f(x) = \sqrt{x}$ and $L(x) = \frac{1}{20}(x - 100) + 10$ on Desmos and zoom in near $x = 100$.

Check it: graph $f(x) = \sqrt{x}$ with the tangent line



Graph $f(x) = \sqrt{x}$ and $L(x) = \frac{1}{20}(x - 100) + 10$ on Desmos and zoom in near $x = 100$.

Near $x = 100$ the curve and the line are indistinguishable — that is why $L(109)$ is such a good estimate of $\sqrt{109}$. Farther from 100 they separate, and the approximation gets worse.

Try it: approximate $(3.81)^2 + \sqrt{3.81}$

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Set-up

$f(x) = x^2 + \sqrt{x}$ and $a = 4$ (since 3.81 is near 4, and 4 is easy to plug in).

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Use a linear approximation to approximate $(3.81)^2 + \sqrt{3.81}$.

Set-up

$f(x) = x^2 + \sqrt{x}$ and $a = 4$ (since 3.81 is near 4, and 4 is easy to plug in).

1. **Slope:** $f'(x) = 2x + \frac{1}{2}x^{-1/2}$, so $f'(4) = 2(4) + \dfrac{1}{2\sqrt{4}} = 8 + \dfrac{1}{4} = \dfrac{33}{4}$.

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Use a linear approximation to approximate $(3.81)^2 + \sqrt{3.81}$.

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1. **Slope:** $f'(x) = 2x + \frac{1}{2}x^{-1/2}$, so $f'(4) = 2(4) + \dfrac{1}{2\sqrt{4}} = 8 + \dfrac{1}{4} = \dfrac{33}{4}$.
2. **Point of tangency:** $(4, f(4)) = (4, 18)$.

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = m(x - x_1)$$
$$y - 18 = \tfrac{33}{4}(x - 4)$$

$(3.81)^2 + \sqrt{3.81}$, continued

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 18 &= \tfrac{33}{4}(x - 4) \\
L(x) &= \tfrac{33}{4}(x - 4) + 18
\end{aligned}
$$

$(3.81)^2 + \sqrt{3.81}$, continued

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\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 18 &= \tfrac{33}{4}(x - 4) \\
L(x) &= \tfrac{33}{4}(x - 4) + 18
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$$f(3.81) = (3.81)^2 + \sqrt{3.81} \approx L(3.81)$$

$(3.81)^2 + \sqrt{3.81}$, continued

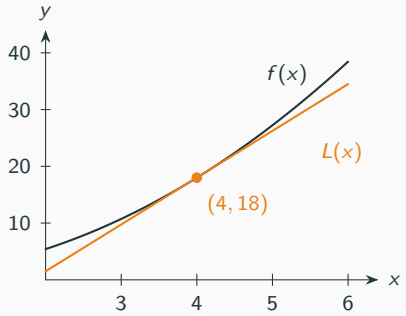
$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 18 &= \tfrac{33}{4}(x - 4) \\
L(x) &= \tfrac{33}{4}(x - 4) + 18
\end{aligned}
$$

$$f(3.81) = (3.81)^2 + \sqrt{3.81} \approx L(3.81)$$

Answer

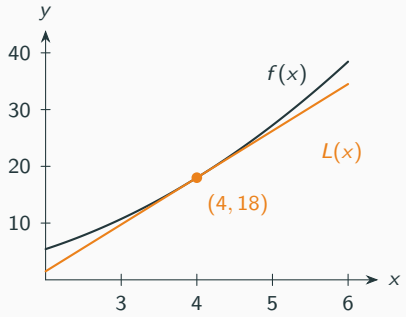
$f(3.81) \approx L(3.81) = \dfrac{33}{4}(3.81 - 4) + 18$ (which is 16.4325)

Check it: graph $f(x) = x^2 + \sqrt{x}$ with the tangent line



Graph $f(x) = x^2 + \sqrt{x}$ and $L(x) = \frac{33}{4}(x - 4) + 18$ on Desmos.

Check it: graph $f(x) = x^2 + \sqrt{x}$ with the tangent line



Graph $f(x) = x^2 + \sqrt{x}$ and $L(x) = \frac{33}{4}(x - 4) + 18$ on Desmos.

At $x = 3.81$ the two are almost on top of each other, so $L(3.81) = 16.4325$ is close to the true value $f(3.81) \approx 16.468$.

Back to the original problem: drug concentration

The concentration of a certain drug in the bloodstream t hours after being administered is approximately $C(t) = \dfrac{5t}{9 + t^2}$.

Try it

Approximate the change in concentration for $t = 1$ to $t = 1.5$. (We want to approximate $C(1.5) - C(1)$.)

Back to the original problem: drug concentration

The concentration of a certain drug in the bloodstream t hours after being administered is approximately $C(t) = \dfrac{5t}{9 + t^2}$.

Try it

Approximate the change in concentration for $t = 1$ to $t = 1.5$. (We want to approximate $C(1.5) - C(1)$.)

Set-up

$C(t) = \dfrac{5t}{9 + t^2}$ and $a = 1$, which is near 1.5.

Drug concentration: slope and point of tangency

1. **Slope** (quotient rule):

$$C'(t) = \frac{5(9 + t^2) - 5t(2t)}{(9 + t^2)^2}$$

Drug concentration: slope and point of tangency

1. **Slope** (quotient rule):

$$C'(t) = \frac{5(9+t^2) - 5t(2t)}{(9+t^2)^2}$$

$$C'(1) = \frac{5(9+1^2) - 5(1)(2\cdot 1)}{(9+1^2)^2} = \frac{5(10) - 5(2)}{10^2} = \frac{50-10}{100} = \frac{40}{100} = \frac{2}{5}$$

Drug concentration: slope and point of tangency

1. **Slope** (quotient rule):

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2. **Point of tangency:** $(1, C(1)) = \left(1, \frac{1}{2}\right)$.

Drug concentration: the linear approximation

$$y - y_1 = m(x - x_1)$$

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$$y - y_1 = m(x - x_1)$$
$$y - \tfrac{1}{2} = \tfrac{2}{5}(x - 1)$$

Drug concentration: the linear approximation

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \tfrac{1}{2} &= \tfrac{2}{5}(x - 1) \\
L(t) &= \tfrac{2}{5}(t - 1) + \tfrac{1}{2} \qquad \text{(the variable in this problem is } t\text{)}
\end{aligned}
$$

Drug concentration: the linear approximation

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\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \tfrac{1}{2} &= \tfrac{2}{5}(x - 1) \\
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\end{aligned}
$$

Now approximate the change

$C(1.5) - C(1) \approx L(1.5) - L(1)$

Drug concentration: the change from $t = 1$ to $t = 1.5$

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Drug concentration: the change from $t = 1$ to $t = 1.5$

$$C(1.5) - C(1) \approx L(1.5) - L(1)$$

$$L(1.5) - L(1) = \frac{2}{5}(1.5 - 1) + \cancel{\frac{1}{2}} - \left(\cancel{\frac{2}{5}(1 - 1)} + \cancel{\frac{1}{2}}\right)$$

Drug concentration: the change from $t = 1$ to $t = 1.5$

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$$= \tfrac{2}{5}\left(\tfrac{1}{2}\right) = \tfrac{2}{10} = \tfrac{1}{5}$$

Drug concentration: the change from $t = 1$ to $t = 1.5$

$$
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C(1.5) - C(1) &\approx L(1.5) - L(1) \\
L(1.5) - L(1) &= \tfrac{2}{5}(1.5 - 1) + \cancel{\tfrac{1}{2}} - \left(\cancel{\tfrac{2}{5}(1 - 1)} + \cancel{\tfrac{1}{2}}\right) \\
&= \tfrac{2}{5}\left(\tfrac{1}{2}\right) = \tfrac{2}{10} = \tfrac{1}{5}
\end{aligned}
$$

Interpretation

From $t = 1$ to $t = 1.5$ hours, the concentration increases by approximately $\frac{1}{5}$.

Example: blood alcohol concentration

The function

$$A(t) = .003631t^3 - .03746t^2 + .1012t + .009$$

gives the approximate blood alcohol concentration in a 170 lb woman t hours after drinking 2 oz of alcohol on an empty stomach, for t in the interval $[0, 5]$.

Try it

(a) Approximate the change in alcohol level from 1 to 1.2 hrs.

(b) Approximate the change in alcohol level from 3 to 3.2 hrs.

BAC (a): from 1 to 1.2 hrs

Take $a = 1$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$

BAC (a): from 1 to 1.2 hrs

Take $a = 1$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$
$$A'(1) = .003631(3)(1)^2 - .03746(2)(1) + .1012 = .037173$$

BAC (a): from 1 to 1.2 hrs

Take $a = 1$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$
$$A'(1) = .003631(3)(1)^2 - .03746(2)(1) + .1012 = .037173$$

Point of tangency: $(1, A(1)) = (1, .076371)$.

BAC (a): from 1 to 1.2 hrs

Take $a = 1$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$
$$A'(1) = .003631(3)(1)^2 - .03746(2)(1) + .1012 = .037173$$

Point of tangency: $(1, A(1)) = (1, .076371)$.

$$L(t) - .076371 = .037173(t - 1)$$
$$L(t) = .037173(t - 1) + .076371$$

BAC (a), continued

$$A(1.2) - A(1) \approx L(1.2) - L(1)$$

BAC (a), continued

$$A(1.2) - A(1) \approx L(1.2) - L(1)$$

$$L(1.2) - L(1) = .037173(1.2 - 1) + \cancel{.076371} - \left(\cancel{.037173(1 - 1) + .076371}\right)$$

BAC (a), continued

$$
\begin{aligned}
A(1.2) - A(1) &\approx L(1.2) - L(1) \\
L(1.2) - L(1) &= .037173(1.2 - 1) + .076371 - \big(.037173(1 - 1) + .076371\big) \\
&= .037173(.2)
\end{aligned}
$$

BAC (a), continued

$$
\begin{aligned}
A(1.2) - A(1) &\approx L(1.2) - L(1) \\
L(1.2) - L(1) &= .037173(1.2 - 1) + \cancel{.076371} - \left(\cancel{.037173(1 - 1)} + \cancel{.076371}\right) \\
&= .037173(.2) \\
&= .0074346
\end{aligned}
$$

BAC (a), continued

$$
\begin{aligned}
A(1.2) - A(1) &\approx L(1.2) - L(1) \\
L(1.2) - L(1) &= .037173(1.2 - 1) + .076371 - \big(.037173(1 - 1) + .076371\big) \\
&= .037173(.2) \\
&= .0074346
\end{aligned}
$$

Interpretation

Positive, so the BAC is **increasing**: between 1 hr and 1.2 hrs after drinking, the BAC rises by about .0074346.

BAC (b): from 3 to 3.2 hrs

Take $a = 3$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$

BAC (b): from 3 to 3.2 hrs

Take $a = 3$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$
$$A'(3) = .003631(3)(3)^2 - .03746(2)(3) + .1012 = -.025523$$

BAC (b): from 3 to 3.2 hrs

Take $a = 3$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$
$$A'(3) = .003631(3)(3)^2 - .03746(2)(3) + .1012 = -.025523$$

Point of tangency: $(3, A(3)) = (3, .073497)$.

BAC (b): from 3 to 3.2 hrs

Take $a = 3$.

$$A'(t) = .003631(3)t^2 - .03746(2)t + .1012$$
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Point of tangency: $(3, A(3)) = (3, .073497)$.

$$L(t) - .073497 = -.025523(t - 3)$$
$$L(t) = -.025523(t - 3) + .073497$$

BAC (b), continued

$$A(3.2) - A(3) \approx L(3.2) - L(3)$$

BAC (b), continued

$$A(3.2) - A(3) \approx L(3.2) - L(3)$$

$$L(3.2) - L(3) = -.025523(3.2 - 3) + .073497 - \left(-.025523(3 - 3) + .073497\right)$$

BAC (b), continued

$$
\begin{aligned}
A(3.2) - A(3) &\approx L(3.2) - L(3) \\
L(3.2) - L(3) &= -.025523(3.2 - 3) + .073497 - \big(-.025523(3 - 3) + .073497\big) \\
&= -.025523(.2)
\end{aligned}
$$

BAC (b), continued

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L(3.2) - L(3) &= -.025523(3.2 - 3) + .073497 - \big(-.025523(3 - 3) + .073497\big) \\
&= -.025523(.2) \\
&= -.0051046
\end{aligned}
$$

BAC (b), continued

$$
\begin{aligned}
A(3.2) - A(3) &\approx L(3.2) - L(3) \\
L(3.2) - L(3) &= -.025523(3.2 - 3) + .073497 - \big(-.025523(3 - 3) + .073497\big) \\
&= -.025523(.2) \\
&= -.0051046
\end{aligned}
$$

Interpretation

Negative, so the BAC is **decreasing**: between 3 hrs and 3.2 hrs after drinking, the BAC drops by about .0051046.

Example: blood glucose

The blood glucose level of a patient (measured in mg/dL) is approximated by

$$G(t) = 90 + 40e^{-.5t},$$

where t is measured in hours after a meal.

Try it

Use a linear approximation to estimate the glucose level at $t = 2.1$ hrs.

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where t is measured in hours after a meal.

Try it

Use a linear approximation to estimate the glucose level at $t = 2.1$ hrs.

Take $a = 2$ (near 2.1).

$$G'(t) = -\left(\tfrac{1}{2}\right) 40e^{-.5t}$$

Example: blood glucose

The blood glucose level of a patient (measured in mg/dL) is approximated by

$$G(t) = 90 + 40e^{-.5t},$$

where t is measured in hours after a meal.

Try it

Use a linear approximation to estimate the glucose level at $t = 2.1$ hrs.

Take $a = 2$ (near 2.1).

$$G'(t) = -\left(\tfrac{1}{2}\right) 40e^{-.5t}$$

$$G'(2) = -20e^{-\frac{1}{2}(2)} = -20e^{-1} = -\frac{20}{e}$$

Blood glucose, continued

Point of tangency: $(2, G(2)) = \left(2, 90 + \frac{40}{e}\right)$, so $L(t) = -\frac{20}{e}(t-2) + 90 + \frac{40}{e}$.

Blood glucose, continued

Point of tangency: $(2, G(2)) = \left(2, 90 + \frac{40}{e}\right)$, so $L(t) = -\frac{20}{e}(t-2) + 90 + \frac{40}{e}$.

$$L(2.1) = -\frac{20}{e}(2.1-2) + 90 + \frac{40}{e}$$

Blood glucose, continued

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$$\begin{aligned} L(2.1) &= -\frac{20}{e}(2.1-2) + 90 + \frac{40}{e} \\ &= -\frac{20}{e}(.1) + 90 + \frac{40}{e} \end{aligned}$$

Blood glucose, continued

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$$\begin{aligned} L(2.1) &= -\frac{20}{e}(2.1-2) + 90 + \frac{40}{e} \\ &= -\frac{20}{e}(.1) + 90 + \frac{40}{e} \\ &\approx 103.98 \text{ mg/dL} \end{aligned}$$

Blood glucose, continued

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$$\begin{aligned} L(2.1) &= -\frac{20}{e}(2.1-2) + 90 + \frac{40}{e} \\ &= -\frac{20}{e}(.1) + 90 + \frac{40}{e} \\ &\approx 103.98 \text{ mg/dL} \end{aligned}$$

Interpretation

About 2.1 hours after the meal, the blood glucose level is approximately 103.98 mg/dL.

Summary

Linear approximation of f at $x = a$

$L(x) = f'(a)(x - a) + f(a)$, and $f(x) \approx L(x)$ for x near a.

- $L(x)$ is just the tangent line at $(a, f(a))$: slope $f'(a)$, point $(a, f(a))$, then $y - y_1 = m(x - x_1)$.
- Choose a **near** the x-value you care about *and* easy to evaluate ($a = 100$ for $\sqrt{109}$; $a = 4$ for 3.81).
- To approximate a **change** $f(b) - f(a)$, use $L(b) - L(a) = f'(a)(b - a)$; the constant $f(a)$ cancels. Positive means the quantity is increasing, negative means decreasing.
- Near the point of tangency the curve and the line are indistinguishable; farther away the approximation gets worse.