Sections 3.2 & 3.4: Continuity

What it means to be continuous at a point, and how continuity relates to the derivative

Definition of continuity

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A function f is continuous at $x = c$ if the following 3 conditions are satisfied:

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

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In words: the function has a value at c, the graph is heading somewhere as $x \to c$ (from **both** sides), and it is heading to **that value**.

Example: modeling the growth of a fungus

As a fungus grows its rate of growth changes. Young fungi grow exponentially, while in larger fungi growth slows, and the total dimensions of the fungus increase as a linear function of time. You want to build a mathematical model that describes the two phases of growth. If $R(t)$ is the rate of growth given as a function of time t, then you model

$$R(t) = \begin{cases} 2e^t & \text{if } 0 \leq t \leq t_c \\ a & \text{if } t > t_c \end{cases}$$

where t_c is the time at which the fungus switches from exponential to linear growth and a is a constant.

Fungus (a): for what value of a is $R(t)$ continuous at $t = t_c$?

Try it

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So

$a = 2e^{t_c}$

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$$R(t) = \begin{cases} 2e^t & 0 \le t \le 2 \\ a = 2e^2 & t > 2 \end{cases}$$

Continuous at 2 if $a = 2e^2$.

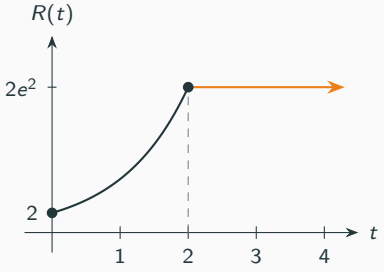
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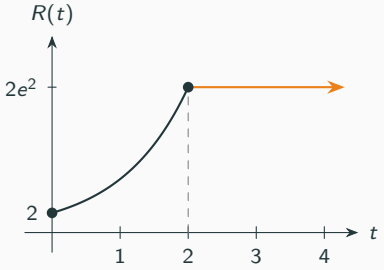
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Continuous at 2 if $a = 2e^2$.

$\lim_{t \to 2^-} R(t) = \lim_{t \to 2^+} R(t) = R(2)$ ✓✓



This isn't always the case!

Example: a graph with $\lim_{x\to 2} f(x) \neq f(2)$

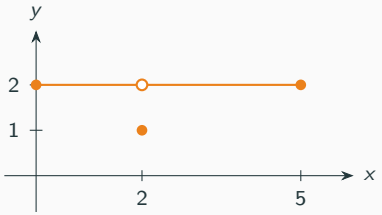
Try it

Draw the graph of $y = f(x)$ with domain $[0, 5]$ with (a) $\lim_{x\to 2} f(x) \neq f(2)$.

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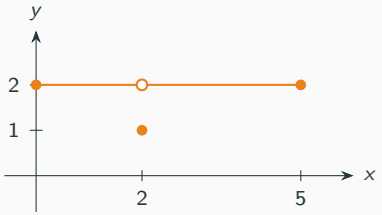
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Removable discontinuity

The limit exists ($= 2$) but $f(2) = 1$: condition 3 fails. Moving one point would fix it.

A formula with a removable discontinuity

Try it

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$$f(x) = \begin{cases} \dfrac{(x-2)(x+2)}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

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$$f(x) = \begin{cases} \dfrac{(x-2)(x+2)}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

For $x \neq 2$ the factors cancel and $f(x) = x + 2$, so $\lim_{x \to 2} f(x) = 4$; but $f(2) = 1$.

Example: a graph with $\lim_{x\to 2} f(x)$ DNE but $\lim_{x\to 2^+} f(x) = f(2)$

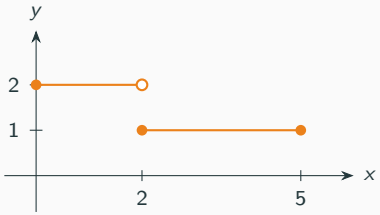
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Draw the graph of $y = f(x)$ with domain $[0, 5]$ with (b) $\lim_{x\to 2} f(x)$ DNE but $\lim_{x\to 2^+} f(x) = f(2)$.

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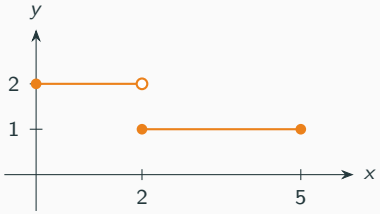
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Jump discontinuity

The one-sided limits are different (2 from the left, 1 from the right), so condition 2 fails.

A formula with a jump discontinuity

Try it

Give a specific $y = f(x)$ with a jump discontinuity at $x = 2$ with domain $(-\infty, \infty)$.

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Give a specific $y = f(x)$ with a jump discontinuity at $x = 2$ with domain $(-\infty, \infty)$.

$$f(x) = \begin{cases} \dfrac{(x-2)(x+2)}{x-2} & x < 2 \\ 1 & x \geq 2 \end{cases}$$

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$$f(x) = \begin{cases} \dfrac{(x-2)(x+2)}{x-2} & x < 2 \\ 1 & x \geq 2 \end{cases}$$

From the left $f(x) = x + 2 \to 4$; from the right $f(x) = 1$. Different one-sided limits: a jump.

Example: choose a and b to make f continuous, part (a)

Example. Find the values of a and b that make $f(x)$ continuous for all x.

$$\text{(a)} \quad f(x) = \begin{cases} -3x + ax^2 & \text{if } x < 1 \\ b & \text{if } x = 1 \\ 4ax - 1 & \text{if } x > 1 \end{cases}$$

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Each piece is a polynomial, so the only place f can fail to be continuous is $x = 1$.
Find a and b that force $f(x)$ to be continuous at $x = 1$.

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Try it

Each piece is a polynomial, so the only place f can fail to be continuous is $x = 1$.
Find a and b that force $f(x)$ to be continuous at $x = 1$.

1. $f(1)$: $\quad f(1) = b$ ✓

Part (a), continued: the one-sided limits

2. $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$:

$$\lim_{x \to 1^-} \left(-3x + ax^2\right) = -3(1) + a(1)^2 = -3 + a \qquad \text{LHL } \checkmark\checkmark$$

Part (a), continued: the one-sided limits

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$$\lim_{x\to 1^+} (4ax - 1) = 4a(1) - 1 = 4a - 1 \qquad \text{RHL } \checkmark\checkmark$$

3. $f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$:

$$b = -3 + a = 4a - 1 \qquad \checkmark\checkmark\checkmark$$

Part (a), continued: now solve for a and b

From $-3 + a = 4a - 1$, then from $b = -3 + a$:

$$-3 + a = 4a - 1$$

Part (a), continued: now solve for a and b

From $-3 + a = 4a - 1$, then from $b = -3 + a$:

$$-3 + a = 4a - 1$$

$$-2 = 3a \qquad \text{(add 1 and subtract } a \text{ on both sides)}$$

Part (a), continued: now solve for a and b

From $-3 + a = 4a - 1$, then from $b = -3 + a$:

$$\begin{aligned} -3 + a &= 4a - 1 \\ -2 &= 3a \qquad \text{(add 1 and subtract } a \text{ on both sides)} \\ -\tfrac{2}{3} &= a \end{aligned}$$

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$$-\tfrac{2}{3} = a$$

$$b = -3 + a = -3 - \tfrac{2}{3} = -\tfrac{9}{3} - \tfrac{2}{3}$$

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$$b = -\tfrac{11}{3}$$

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$$
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-2 &= 3a \qquad \text{(add 1 and subtract } a \text{ on both sides)} \\
-\tfrac{2}{3} &= a \\
b &= -3 + a = -3 - \tfrac{2}{3} = -\tfrac{9}{3} - \tfrac{2}{3} \\
b &= -\tfrac{11}{3}
\end{aligned}
$$

Answer

$a = -\frac{2}{3}$ and $b = -\frac{11}{3}$ make f continuous for all x.

Part (b): continuity at two points at once

$$\text{(b)} \quad f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ ax^2 + 2x + b & \text{if } -1 < x \leq 2 \\ 14 - ax & \text{if } x > 2 \end{cases}$$

Try it

Find a and b that force $f(x)$ to be continuous at -1 and 2 **simultaneously**.

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The formula changes at $x = -1$ and at $x = 2$, so we check the three conditions at each.

Part (b), continued: at $x = -1$

At $x = -1$:

1. $f(-1) = 3$ ✓

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1. $f(-1) = 3$ ✓
2. $\lim_{x \to -1^-} 3 = 3$ (LHL) ✓✓

   $\lim_{x \to -1^+} \left(ax^2 + 2x + b\right) \overset{\text{DSP}}{=} a(-1)^2 + 2(-1) + b = a - 2 + b$ (RHL) ✓✓

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3. $f(-1) = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$: $\quad 3 = 3 = a - 2 + b$ ✓✓✓

Part (b), continued: at $x = -1$

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3. $f(-1) = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$: $3 = 3 = a - 2 + b$ ✓✓✓

Note

DSP = direct substitution property: a polynomial's limit is found by plugging in.

NOT possible to solve for a or b from this equation alone.

Part (b), continued: at $x = 2$

At $x = 2$:

1. $f(2) = a(2)^2 + 2(2) + b = 4a + 4 + b$ ✓

Part (b), continued: at $x = 2$

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NOT possible to solve for a or b from this alone either — but now we have **two** equations in a and b.

Part (b), continued: a system of two equations

$$3 = a - 2 + b \xrightarrow{\text{solve for } b} b = 5 - a$$

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$$4a + 4 + b = 14 - 2a \qquad (\text{substitute } b = 5 - a)$$

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$$4a + 4 + 5 - a = 14 - 2a$$

$$3a + 9 = 14 - 2a$$

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$$a = 1, \qquad b = 5 - 1 = 4$$

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$$a = 1, \qquad b = 5 - 1 = 4$$

Answer

$a = 1$ and $b = 4$ make f continuous at both $x = -1$ and $x = 2$, hence for all x.

The derivative and continuity

How are the derivative and continuity related?

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Continuity is necessary for differentiability

If $f(x)$ is **NOT** continuous at $x = c$, then $f'(c)$ DNE.

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If $f(x)$ is **NOT** continuous at $x = c$, then $f'(c)$ DNE.

But continuity is NOT sufficient for differentiability

$f(x)$ could be continuous at $x = c$, but $f'(c)$ still might **NOT** exist.

The derivative and continuity

How are the derivative and continuity related?

Continuity is necessary for differentiability

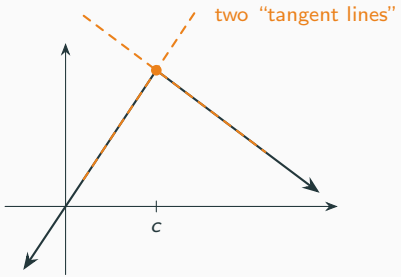
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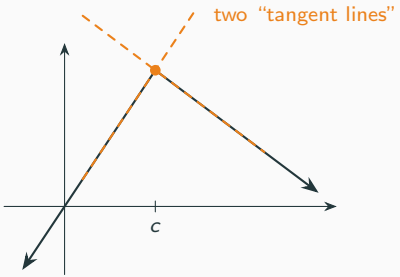
So when is $f(x)$ continuous at $x = c$ but $f'(c)$ DNE? There are three pictures to know.

Continuous but not differentiable: (1) a corner



(1) $f(x)$ has a corner at $x = c$.

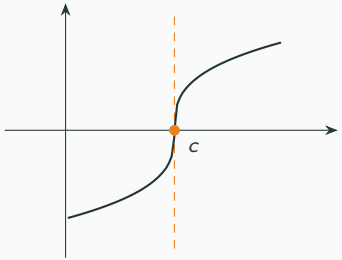
Continuous but not differentiable: (1) a corner



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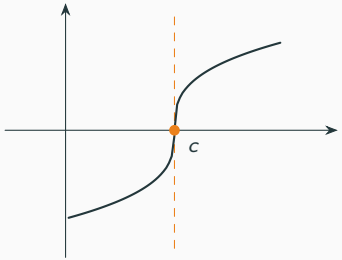
There are 2 different ways to draw the tangent line at the corner — but tangent lines are **UNIQUE**, so there is **no** tangent line there. $f'(c)$ DNE.

Continuous but not differentiable: (2) a vertical tangent



(2) $f(x)$ has a vertical tangent at $x = c$.

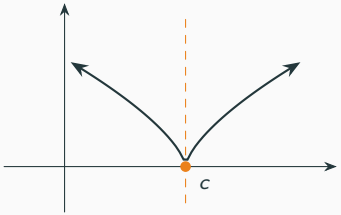
Continuous but not differentiable: (2) a vertical tangent



(2) $f(x)$ has a vertical tangent at $x = c$.

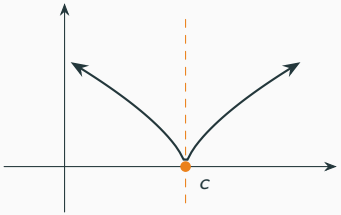
The tangent line at $x = c$ is a vertical line, which has **undefined slope** — so $f'(c)$ DNE.

Continuous but not differentiable: (3) a cusp



(3) $f(x)$ has a cusp at $x = c$.

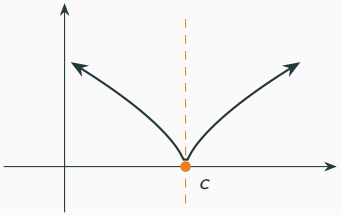
Continuous but not differentiable: (3) a cusp



(3) $f(x)$ has a cusp at $x = c$.

The tangent line at a cusp is vertical, so $f'(c)$ DNE.

Continuous but not differentiable: (3) a cusp



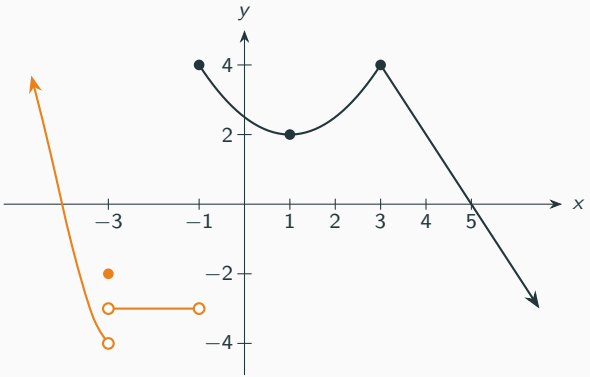
(3) $f(x)$ has a cusp at $x = c$.

The tangent line at a cusp is vertical, so $f'(c)$ DNE.

Summary of the three

corner, vertical tangent, cusp: continuous at c, but no (usable) tangent line, so no derivative.

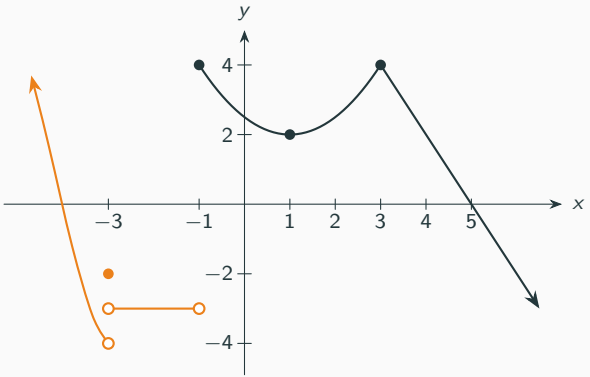
Example: reading continuity and the derivative from a graph



Example. Given $y = f(x)$, find the following.

(a) List all x-values where $f(x)$ is NOT continuous.

Example: reading continuity and the derivative from a graph



Example. Given $y = f(x)$, find the following.

(a) List all x-values where $f(x)$ is NOT continuous.

(a)

$x = -3, -1$ (a break in the graph at each)

Graph example, continued: where $f'(x)$ DNE and where $f'(x) = 0$

(b) List all x-values where $f'(x)$ DNE.

Graph example, continued: where $f'(x)$ DNE and where $f'(x) = 0$

(b) List all x-values where $f'(x)$ DNE.

(b)

$x = 3$ (corner), $\quad x = -3, -1$ (not continuous)

Graph example, continued: where $f'(x)$ DNE and where $f'(x) = 0$

(b) List all x-values where $f'(x)$ DNE.

(b)

$x = 3$ (corner), $\quad x = -3, -1$ (not continuous)

(c) State where $f'(x) = 0$ on $(-5, 5)$ — all x where the tangent line is **horizontal**.

Graph example, continued: where $f'(x)$ DNE and where $f'(x) = 0$

(b) List all x-values where $f'(x)$ DNE.

(b)

$x = 3$ (corner), $x = -3, -1$ (not continuous)

(c) State where $f'(x) = 0$ on $(-5, 5)$ — all x where the tangent line is **horizontal**.

(c)

$x = 1$ (the vertex of the upward-opening parabola), and all x in the interval $(-3, -1)$ (the graph itself is horizontal there).

Graph example, continued: the sign of $f'(x)$

(d) State where $f'(x) > 0$ on $(-5, 5)$ — where $f(x)$ is **increasing**.

Graph example, continued: the sign of $f'(x)$

(d) State where $f'(x) > 0$ on $(-5, 5)$ — where $f(x)$ is **increasing**.

(d)

all x in the interval $(1, 3)$

Graph example, continued: the sign of $f'(x)$

(d) State where $f'(x) > 0$ on $(-5, 5)$ — where $f(x)$ is **increasing**.

(d)

all x in the interval $(1, 3)$

(e) State where $f'(x) < 0$ on $(-5, 5)$ — where $f(x)$ is **decreasing**.

Graph example, continued: the sign of $f'(x)$

(d) State where $f'(x) > 0$ on $(-5, 5)$ — where $f(x)$ is **increasing**.

(d)

all x in the interval $(1, 3)$

(e) State where $f'(x) < 0$ on $(-5, 5)$ — where $f(x)$ is **decreasing**.

(e)

all x in the intervals $(-5, -3)$, $(-1, 1)$, $(3, 5)$

Summary

Continuous at $x = c$

(1) $f(c)$ is defined, (2) $\lim_{x \to c} f(x)$ exists (LHL = RHL), (3) $\lim_{x \to c} f(x) = f(c)$.

- For a piecewise function, check the three conditions at each place the formula changes. Unknown constants: set $f(c) = \text{LHL} = \text{RHL}$ and solve; two break points give a system of two equations.
- Ways to fail: a *removable* discontinuity (limit exists, wrong or missing value) or a *jump* (LHL $\neq$ RHL).
- Not continuous at $c \Rightarrow f'(c)$ DNE. Continuous at c does **not** guarantee $f'(c)$ exists: a corner, a vertical tangent, or a cusp has no derivative there.
- On a graph: $f'(x) = 0$ where the tangent line is horizontal, $f'(x) > 0$ where f increases, $f'(x) < 0$ where f decreases.