Section 7.4: The Fundamental Theorem of Calculus

Definite integrals from antiderivatives

Two kinds of integrals

The indefinite integral

$$\int f(x)\,dx$$

is the **most general antiderivative** of $f(x)$ with respect to x: a family of functions, $F(x) + C$.

The definite integral

$$\int_a^b f(x)\,dx$$

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They are related by the following theorem. It gives a much faster way to calculate a definite integral than rectangles or geometry.

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Let f be continuous on $[a, b]$, and let F be **any** antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

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Why "any" antiderivative? With $F(x) + C$ instead, $\big(F(b) + C\big) - \big(F(a) + C\big) = F(b) - F(a)$. The C cancels, so we leave it out.

Example 1: building a strategy

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$$\frac{8x^2 - 7}{x^3} = \frac{8x^2}{x^3} - \frac{7}{x^3} = \frac{8}{x} - 7x^{-3}$$

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Strategy, Step 1: find an antiderivative

Find an antiderivative $F(x)$ of the integrand with the indefinite integral strategy: rewrite, split, use the rules. Leave out the $+C$.

Step 2: evaluate $F(b) - F(a)$

$$\int_1^5 \left(\frac{8}{x} - 7x^{-3}\right)\, dx = \left(8 \ln|x| + \frac{7}{2x^2}\right)\Bigg|_1^5$$

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Substitute the upper limit 5, then subtract the value at the lower limit 1:

$$= \left(8\ln(5) + \frac{7}{2(5)^2}\right) - \left(8\ln(1) + \frac{7}{2(1)^2}\right)$$

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Strategy, Step 2: evaluate

Write $F(x)|_a^b$, then compute $F(b) - F(a)$: the upper limit first, minus the lower limit. Put each value in parentheses.

Step 3: simplify

Since $\ln(1) = 0$:

$$\int_1^5 \frac{8x^2 - 7}{x^3}\,dx = 8\ln(5) + \frac{7}{50} - \frac{7}{2} \approx 9.52$$

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Strategy, Step 3: simplify

Simplify, using facts such as $\ln(1) = 0$ and $e^0 = 1$. Give a decimal approximation if it is asked for.

The strategy we just built

Calculating $\int_a^b f(x)\,dx$ with the Fundamental Theorem, Part 1

1. **Find an antiderivative** $F(x)$: rewrite, split, use the rules. No $+ C$.
2. **Evaluate** $F(x)\big|_a^b = F(b) - F(a)$: the upper limit first, minus the lower limit, each value in parentheses.
3. **Simplify.**

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2. **Evaluate** $F(x)\big|_a^b = F(b) - F(a)$: the upper limit first, minus the lower limit, each value in parentheses.
3. **Simplify.**

Compare with Sections 7.3 and 7.4: no rectangles and no geometry, only an antiderivative.

Example 2: substitution in a definite integral

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$u = 4x^3+2, \quad \frac{du}{dx} = 12x^2, \quad du = 12x^2\,dx, \quad$ so $\frac{1}{12}\,du = x^2\,dx.$

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The limits 0 and 1 are values of x, not of u. There are **two ways** to finish.

Example 2, Option 1: new limits of integration

Option 1: change the limits to values of u, using $u = 4x^3 + 2$:

$x = 0$: $u = 4(0)^3 + 2 = 2$; $x = 1$: $u = 4(1)^3 + 2 = 6$.

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Strategy, Option 1: new limits

Put each limit of integration into $u = u(x)$ to get the new limits. Integrate in u and evaluate at the new limits: there is no need to go back to x.

Example 2, Option 2: back to x

Option 2: find the indefinite integral in u, then go back to x:

$$\frac{7}{12}\int u^{-1}\,du = \frac{7}{12}\ln|u| = \frac{7}{12}\ln\left|4x^3+2\right| \qquad \text{(back to } x\text{)}$$

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Then evaluate at the original limits:

$$\int_0^1 7x^2\left(4x^3+2\right)^{-1}\,dx = \frac{7}{12}\ln\left|4x^3+2\right|\Big|_0^1 = \frac{7}{12}\big(\ln(6)-\ln(2)\big)$$

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Strategy, Option 2: back to x

Find the indefinite integral in u, substitute back to x, then evaluate at the original limits of integration.

The strategy we just built

Substitution in a definite integral

1. **Choose u and find du**, as in Section 7.2 (fix a constant factor if needed).
2. **Finish in one of two ways:**
   - **Option 1 (new limits):** change the limits to $u(a)$ and $u(b)$, integrate in u, and evaluate at the new limits.
   - **Option 2 (back to x):** integrate in u, substitute back to x, and evaluate at the original limits a and b.

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The limits must match the variable: u-limits go with u, and x-limits go with x.

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Careful: the factor $\frac{5}{9}$ multiplies *both* logarithms.

Try it — Example 4: a piecewise function

Example 4. Calculate $\displaystyle\int_0^{10} g(x)\,dx$, where $g(x) = \begin{cases} 3x, & 0 \le x \le 2 \\ -5x + 10, & 2 < x \le 3 \\ -5, & x > 3 \end{cases}$

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Split the integral where the formula changes, at $x = 2$ and $x = 3$ (property 4 of definite integrals):

$$\int_0^{10} g(x)\,dx = \int_0^{2} 3x\,dx + \int_2^{3} (-5x + 10)\,dx + \int_3^{10} (-5)\,dx$$

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$$= \frac{3x^2}{2}\bigg|_0^2 + \left(-\frac{5x^2}{2} + 10x\right)\bigg|_2^3 + (-5x)|_3^{10}$$

Example 4, continued

$$= \left(\frac{3}{2}(2)^2 - \frac{3}{2}(0)^2\right) + \left(-\frac{5}{2}(3)^2 + 10(3) + \frac{5}{2}(2)^2 - 10(2)\right) + (-5(10) + 5(3))$$

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$$= \frac{12}{2} - \frac{45}{2} + 30 + \frac{20}{2} - 20 - 50 + 15 = -\frac{13}{2} - 25 = -\frac{63}{2}$$

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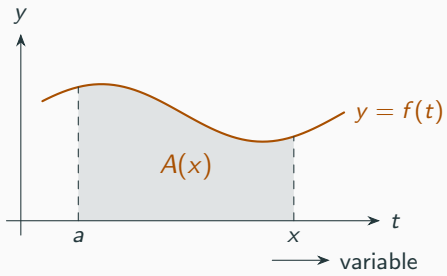
The same answer we got using geometry

In Sections 7.3 and 7.4 (Example 6) we found the same integral from areas: $6 - \frac{5}{2} - 35 = -\frac{63}{2}$.

The area function

Let f be continuous. The **area function** $A(x)$ gives the area under the curve $y = f(t)$ from a to x:

$$A(x) = \int_a^x f(t)\,dt$$

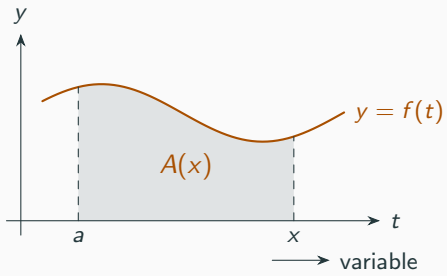


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The upper limit x is the **variable**: as x moves to the right, more area is included. The letter t is the variable of integration.



The Fundamental Theorem of Calculus, Part 2

If $f(t)$ is continuous on $[a, b]$ and $A(x) = \int_a^x f(t)\,dt$ for $a \leq x \leq b$, then A is continuous on $[a, b]$, differentiable on (a, b), and

$$\frac{d}{dx}\left[A(x)\right] = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \qquad \text{for all } x \text{ in } (a, b).$$

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In words: the area grows at a rate equal to the height of the curve at x.

Example 5

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The integrand $f(t) = \sqrt{t^3 + 1}$ is continuous for $t \geq 1$, so Part 2 applies: replace t by x in the integrand.

Answer

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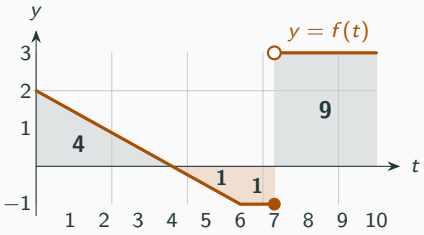
We never needed an antiderivative of $\sqrt{t^3 + 1}$.

Example 6: an area function from a graph

Example 6. Let $F(x) = \int_0^x f(t)\,dt$, where $y = f(t)$ is graphed. The areas between the graph and the t-axis are 4 and 9 above the axis, and 1 and 1 below it. Find:

(a) $F(10)$

(b) $F'(6)$

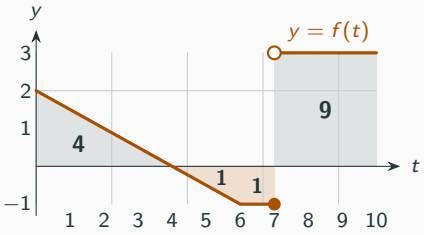


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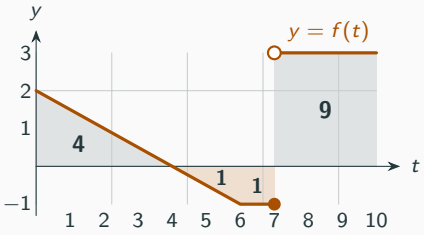
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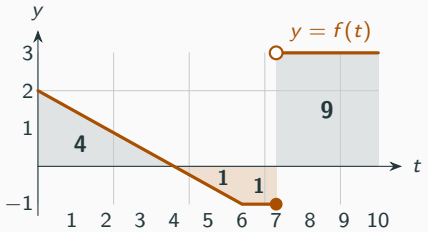
(a) $F(10) = \int_0^{10} f(t)\,dt = 4 - 1 - 1 + 9 = 11$

(b) By Part 2, $F'(x) = f(x)$, so $F'(6) = f(6) = -1$.

Example 6, continued: two more integrals

(c) $\displaystyle\int_0^6 |f(t)|\,dt$

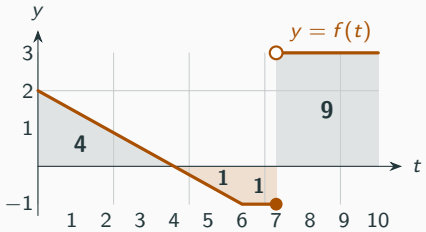
(d) $\displaystyle\int_0^4 \big(f'(t)+5\big)\,dt$



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(c) $\displaystyle\int_0^6 |f(t)|\,dt = 4 + 1 = 5$: every area counts as positive.

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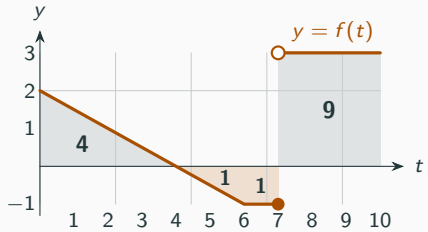


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(d) $\displaystyle\int_0^4 \big(f'(t) + 5\big)\,dt$

$$= \int_0^4 f'(t)\,dt + \int_0^4 5\,dt = f(t)\big|_0^4 + 5t\big|_0^4$$



In (d), f is an antiderivative of f', so Part 1 applies.

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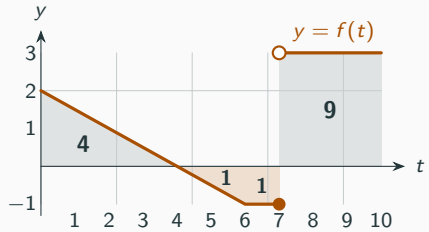
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(d) $\displaystyle\int_0^4 \big(f'(t) + 5\big)\,dt$

$$= \int_0^4 f'(t)\,dt + \int_0^4 5\,dt = f(t)\big|_0^4 + 5t\big|_0^4$$

$$= f(4) - f(0) + 5(4) - 5(0)$$

$$= 0 - 2 + 20 - 0 = 18$$



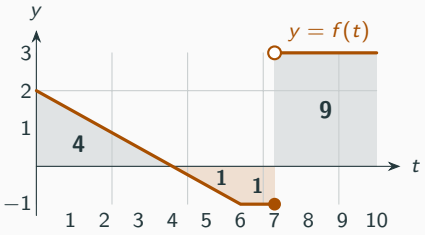
In (d), f is an antiderivative of f', so Part 1 applies.

Example 6, continued: where F increases and decreases

(e) Where is F increasing?

(f) Where is F decreasing?

(g) At what values of x does F have a local max or min?

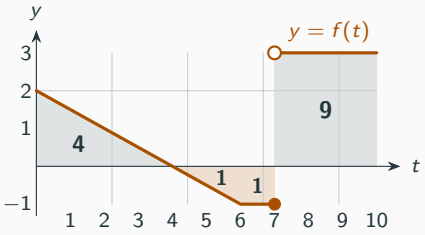


Example 6, continued: where F increases and decreases

(e) Where is F increasing? Wherever $F'(x) = f(x)$ is positive: $(0, 4)$ and $(7, 10)$.

(f) Where is F decreasing?

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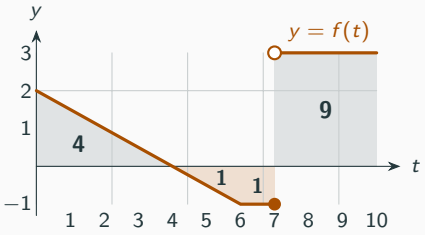


Example 6, continued: where F increases and decreases

(e) Where is F increasing? Wherever $F'(x) = f(x)$ is positive: $(0, 4)$ and $(7, 10)$.

(f) Where is F decreasing? Wherever $F'(x) = f(x)$ is negative: $(4, 7)$.

(g) At what values of x does F have a local max or min?

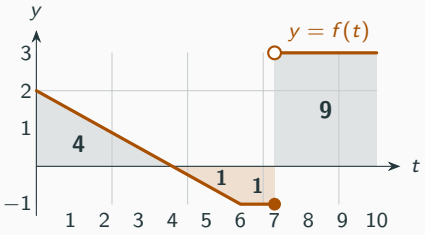


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(g) At what values of x does F have a local max or min? Where $F' = f$ changes sign:



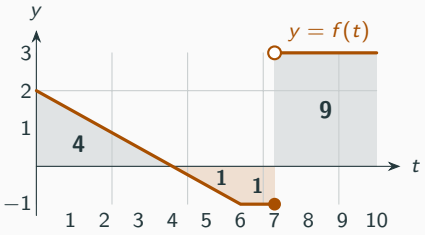
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(g) At what values of x does F have a local max or min? Where $F' = f$ changes sign:

- $x = 4$: f changes from $+$ to $-$, and $F'(4) = 0$: **local max**.



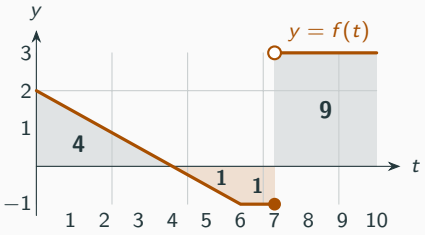
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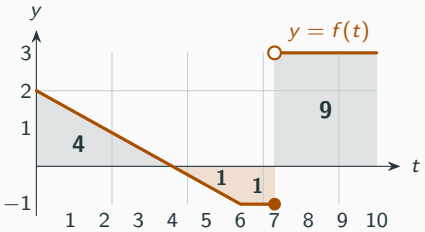
- $x = 4$: f changes from $+$ to $-$, and $F'(4) = 0$: **local max**.
- $x = 7$: f changes from $-$ to $+$. $F'(7)$ does not exist, but F is continuous at $x = 7$: **local min**.



Example 6, continued: concavity

(h) Where is F concave down?

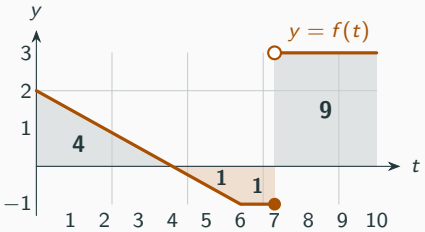
(i) Where is F concave up?



Example 6, continued: concavity

(h) Where is F concave down? Wherever $F' = f$ is decreasing: $(0, 6)$.

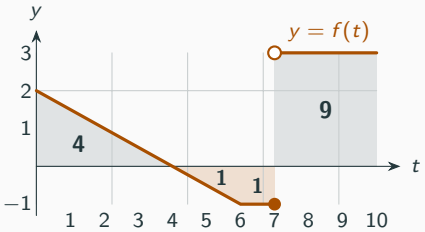
(i) Where is F concave up?



Example 6, continued: concavity

(h) Where is F concave down? Wherever $F' = f$ is decreasing: $(0, 6)$.

(i) Where is F concave up? Wherever $F' = f$ is increasing: none.

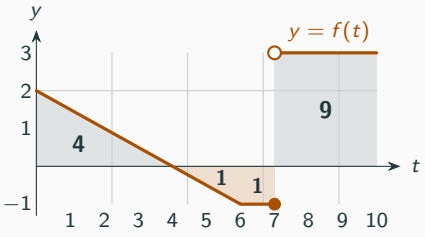


Example 6, continued: concavity

(h) Where is F concave down? Wherever $F' = f$ is decreasing: $(0, 6)$.

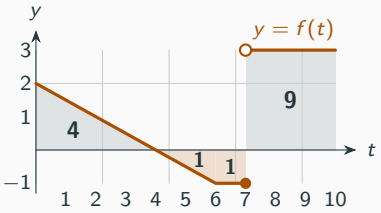
(i) Where is F concave up? Wherever $F' = f$ is increasing: none.

On $(6, 7)$ and $(7, 10)$, f is constant, so the graph of F is a straight line there: neither concave up nor concave down.



Example 6, continued: the absolute max

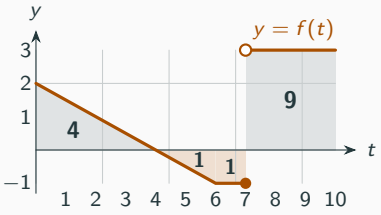
(j) At what x-value does F attain its absolute max on $[0, 10]$?



Example 6, continued: the absolute max

(j) At what x-value does F attain its absolute max on $[0, 10]$?

F is continuous on the closed interval $[0, 10]$, so the EVT applies: compare F at the endpoints and at the critical points $x = 4$ and $x = 7$ from (g).



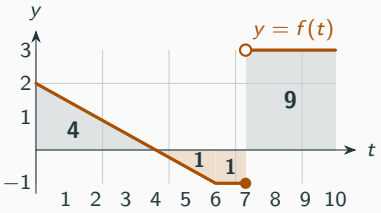
Example 6, continued: the absolute max

(j) At what x-value does F attain its absolute max on $[0, 10]$?

F is continuous on the closed interval $[0, 10]$, so the EVT applies: compare F at the endpoints and at the critical points $x = 4$ and $x = 7$ from (g).

endpoint	$x = 0$:	$F(0) = 0$	$\leftarrow$ **smallest**
endpoint	$x = 10$:	$F(10) = 11$	$\leftarrow$ **largest**
critical point	$x = 4$:	$F(4) = 4$	
critical point	$x = 7$:	$F(7) = 2$	

The absolute max of F is $F(10) = 11$, at $x = 10$.



Reading an area function from the graph of f

What the graph of f tells us about $F(x) = \int_a^x f(t)\,dt$

- **Values:** $F(x)$ is the area from a to x; area below the axis counts as negative.
- **Slope:** $F'(x) = f(x)$ (Part 2), wherever f is continuous.
- **Increasing / decreasing:** where f is positive / negative.
- **Local max / min:** where f changes sign, from $+$ to $-$ / from $-$ to $+$.
- **Concave up / down:** where f is increasing / decreasing.
- **Absolute max and min on $[a, b]$:** the EVT table of endpoints and critical points.

Summary

- **FTC, Part 1:** if F is any antiderivative of a continuous f, then $\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$. Example 1: $8\ln(5) + \frac{7}{50} - \frac{7}{2}$.
- **Strategy:** find an antiderivative (no $+ C$), evaluate $F(b) - F(a)$, simplify.
- **Substitution:** Option 1 changes the limits to u-values; Option 2 goes back to x and uses the original limits. Examples 2 and 3: $\frac{7}{12}\big(\ln(6) - \ln(2)\big)$ and $\frac{5}{9}\big(\ln(5) - \ln(2)\big)$.
- A piecewise integral splits where the formula changes: $\int_0^{10} g(x)\,dx = -\frac{63}{2}$, as with geometry.
- **FTC, Part 2:** $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. Example 5: $A'(x) = \sqrt{x^3 + 1}$.
- For $F(x) = \int_0^x f(t)\,dt$, the graph of $f = F'$ shows where F increases, decreases, has local extrema and is concave up or down.