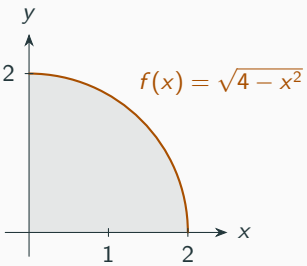
Sections 7.3 and 7.4: Area and the Definite Integral

From rectangles to exact area

Example 1: the area under a quarter circle

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$$f(x) = \sqrt{4 - x^2}.$$



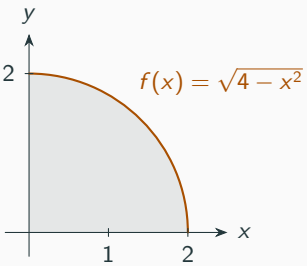
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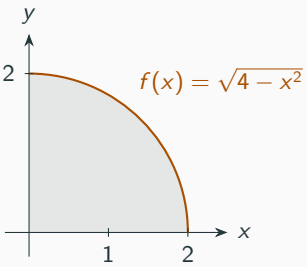
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Most regions are not pieces of circles, so we need another way: approximate the area with rectangles, then make the approximation exact.



(a) Two rectangles: Steps 1 and 2

(a) Approximate the area of the region using 2 rectangles. Determine the height of each rectangle by the value of the function at the left endpoint.

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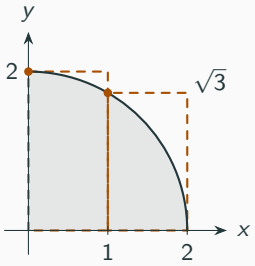
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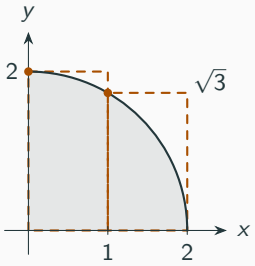


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Strategy, Steps 1 and 2: width and heights

Divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. In each one, choose a point (here, the left endpoint) and find the height of f there.

(a) Step 3: add the areas of the rectangles

Each rectangle has area (height)(width):

$$A \approx 1 \cdot f(0) + 1 \cdot f(1) = 1 \cdot 2 + 1 \cdot \sqrt{3} = 2 + \sqrt{3} \approx 3.73$$

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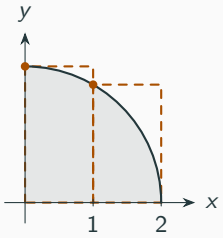
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Is this an overestimate or an underestimate? $3.73 > \pi \approx 3.14$: an **overestimate**. f is decreasing on $[0, 2]$, so each rectangle (its height taken at the left endpoint) sticks out above the curve.

(a) What could we do to get an underestimate?

Use the **right endpoints** instead: each rectangle then lies under the curve, and $1 \cdot f(1) + 1 \cdot f(2) = \sqrt{3} + 0 \approx 1.73$.

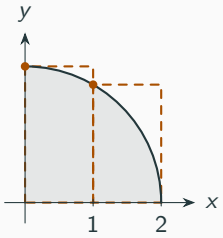
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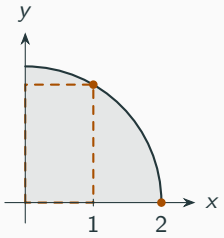
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Right endpoints: $A \approx 1.73$
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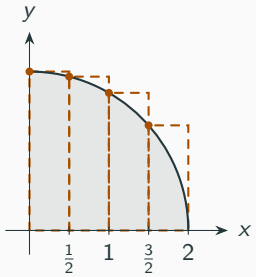
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Steps 2 and 3:

$$A \approx \tfrac{1}{2}f(0) + \tfrac{1}{2}f\left(\tfrac{1}{2}\right) + \tfrac{1}{2}f(1) + \tfrac{1}{2}f\left(\tfrac{3}{2}\right) \approx 3.50$$



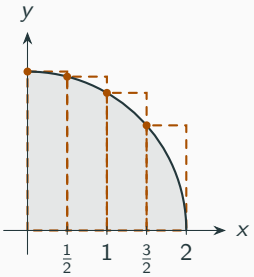
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Which is a better approximation? 3.50, closer to $\pi \approx 3.14$. **What about 8 rectangles?** About 3.34: more rectangles, better approximation.

The strategy we just built

Approximating the area under $y = f(x)$ on $[a, b]$ with rectangles

1. **Divide** $[a, b]$ into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$.
2. **Choose a point** x_i in the ith subinterval (for example, its left endpoint), and find the height $f(x_i)$.
3. **Add the areas** of the rectangles:

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

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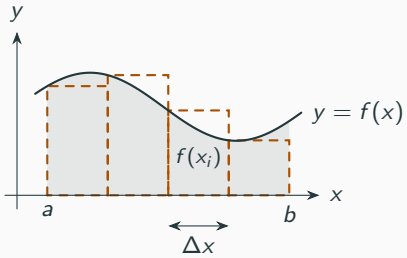
The symbol $\sum$ (sigma) means "add up": $\sum_{i=1}^{n} f(x_i)\Delta x$ is the sum of the n rectangle areas.

From approximate area to exact area

How do we get the **exact** area? Let the number of rectangles go to infinity, so that the width of each subinterval goes to 0:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

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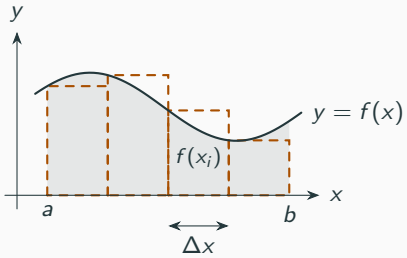
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This limit is called the **definite integral**.



The definite integral

Definition: the definite integral

If f is defined on the interval $[a, b]$, the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

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provided the limit exists.

- b is the **upper limit of integration**, and a is the **lower limit of integration**.
- If $f(x) \geq 0$ on $[a, b]$, the definite integral is the area under the graph of f on $[a, b]$. In Example 1, $\displaystyle\int_0^2 \sqrt{4 - x^2}\,dx = \pi$.

Properties of definite integrals

1. $\int_a^a f(x)\,dx = 0$

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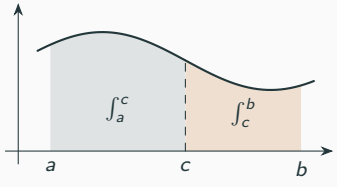
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Properties 2 and 3 are the same as for indefinite integrals (Section 7.1). Property 1 says that a region with no width has no area.

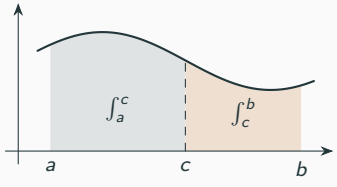
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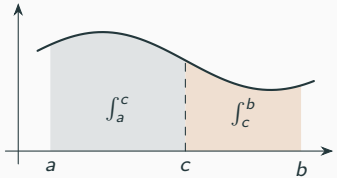
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6. $\displaystyle\int_a^b f(x)\,dx = \underset{\oplus}{(\text{area above the } x\text{-axis})} + \underset{\ominus}{(\text{area below the } x\text{-axis})}$

Area above the x-axis counts as positive ($\oplus$), and area below the x-axis counts as negative ($\ominus$).

Example 2: definite integrals from a graph

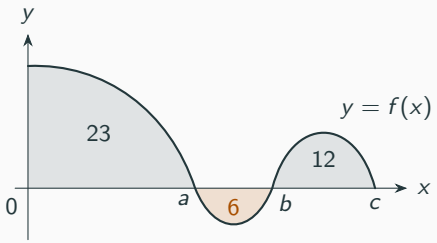
Example 2. The regions between the graph of $y = f(x)$ and the x-axis have the areas shown. Find:

(a) $\displaystyle\int_0^a f(x)\,dx$

(b) $\displaystyle\int_0^b f(x)\,dx$

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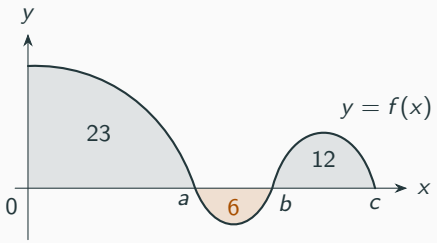
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(a) $\displaystyle\int_0^a f(x)\,dx = 23$

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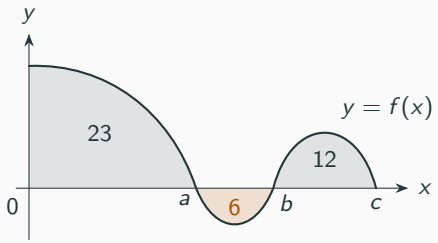
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The area 6 is below the x-axis, so it counts as -6 (property 6).

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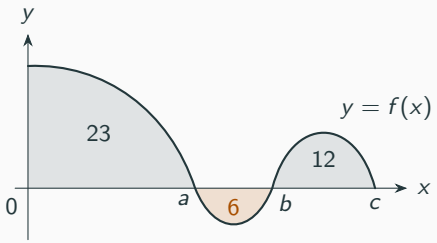
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In (c), the limits are in reverse order (property 5).

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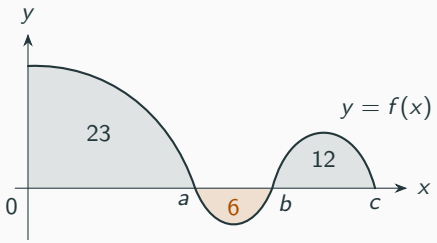
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(d) $\displaystyle\int_a^c f(x)\,dx = -6 + 12 = 6$



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Example 2, continued: absolute values

(e) $\displaystyle\int_0^c |f(x)|\,dx$: the graph of $|f(x)|$ reflects everything below the x-axis to above it, so every area counts as positive:

(f) $\displaystyle\left|\int_0^c f(x)\,dx\right|$: evaluate the integral, *then* take the absolute value of the answer:

(g) $\displaystyle\int_c^a |f(x)|\,dx$: the limits are in reverse order, so use property 5:

Example 2, continued: absolute values

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Compare (e) and (f): $\int |f|$ and $\left|\int f\right|$ are different. Absolute value inside the integral makes every area positive; absolute value outside acts only on the final number.

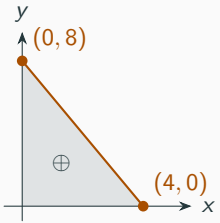
Example 3: a definite integral by geometry

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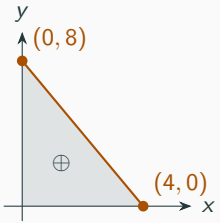
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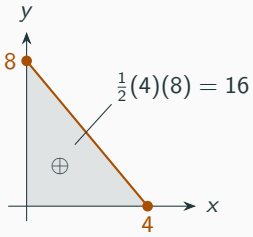
Strategy, Step 1: graph

Graph $y = f(x)$ on $[a, b]$: find the points at the limits of integration, and the x- and y-intercepts.



Example 3, Steps 2–4

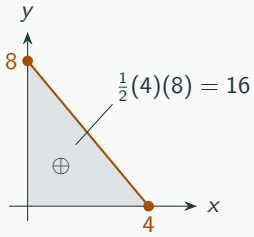
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Step 3: $A = \frac{1}{2}bh = \frac{1}{2}(4)(8) = 16$.

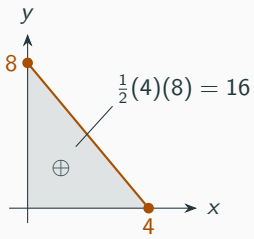


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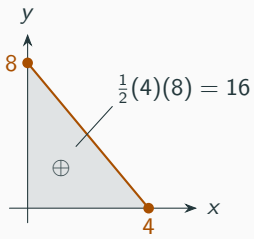


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Strategy, Steps 2–4: shade, measure, add

Shade the region between the graph and the x-axis; split it into triangles and rectangles. Find each area. Add the areas above the x-axis; subtract the areas below.

The strategy we just built

Calculating $\int_a^b f(x)\,dx$ by graphing and geometry

1. **Graph** $y = f(x)$ on $[a, b]$: the points at the limits of integration, and the intercepts.
2. **Shade** the region between the graph and the x-axis, and split it into triangles and rectangles.
3. **Find each area** with geometry: $\frac{1}{2}bh$ for a triangle, bh for a rectangle.
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Step 4 matters as soon as part of the graph is below the x-axis, as in the next example.

Example 4

Example 4. $\int_{-4}^{2} (2x + 4)\, dx$: the area under the graph of $y = 2x + 4$ on $[-4, 2]$.

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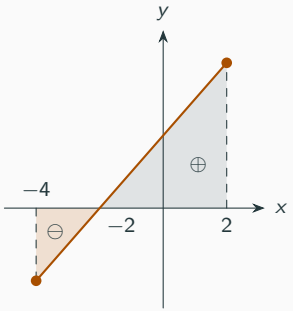
Step 1: at $x = -4$, $y = -4$: the point $(-4, -4)$; at $x = 2$, $y = 8$: the point $(2, 8)$. x-intercept: $0 = 2x + 4$ gives $(-2, 0)$; y-intercept: $(0, 4)$.

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Steps 2 and 3: below the x-axis on $[-4, -2]$, a triangle of area $\frac{1}{2}(2)(4) = 4$; above it on $[-2, 2]$, a triangle of area $\frac{1}{2}(4)(8) = 16$.



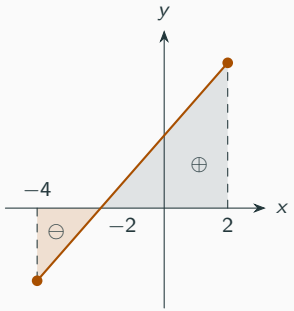
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Step 4: $\int_{-4}^{2}(2x+4)\,dx = -4 + 16 = 12$.



Try it — Example 5

Example 5. $\int_{-5}^{3} \big(|x| - 2\big)\, dx$: the area under the graph of $y = |x| - 2$ on $[-5, 3]$, which is the graph of $y = |x|$ shifted down 2 units.

Try it — Example 5

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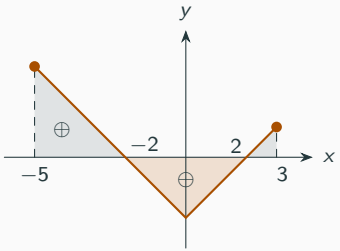
Step 1: the points $(-5, 3)$ and $(3, 1)$; x-intercepts: $|x| = 2$ gives $(-2, 0)$ and $(2, 0)$; y-intercept: $(0, -2)$.

Try it — Example 5

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Steps 2 and 3: three triangles: above on $[-5, -2]$, area $\frac{1}{2}(3)(3) = \frac{9}{2}$; below on $[-2, 2]$, area $\frac{1}{2}(4)(2) = 4$; above on $[2, 3]$, area $\frac{1}{2}(1)(1) = \frac{1}{2}$.



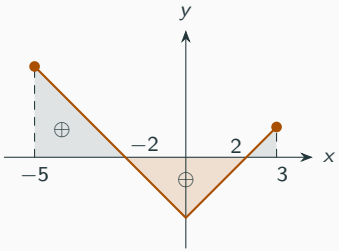
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Step 4: $\displaystyle\int_{-5}^{3} \big(|x| - 2\big)\, dx = \frac{9}{2} - 4 + \frac{1}{2} = 1.$



Try it — Example 6: a piecewise function

Example 6. Calculate $\displaystyle\int_0^{10} g(x)\,dx$, where $g(x) = \begin{cases} 3x, & 0 \le x \le 2 \\ -5x + 10, & 2 < x \le 3 \\ -5, & x > 3 \end{cases}$

Try it — Example 6: a piecewise function

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Step 1: graph each piece.

- $g(x) = 3x$ on $[0, 2]$: a line with positive slope, from $(0, 0)$ to $(2, 6)$.

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- $g(x) = -5x + 10$ on $(2, 3]$: a line with negative slope, from $(2, 0)$, a hole, to $(3, -5)$.

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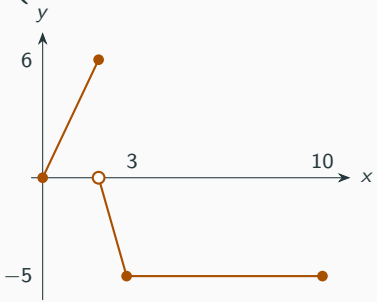
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- $g(x) = -5$ on $(3, 10]$: a horizontal line, from $(3, -5)$ to $(10, -5)$. $(3, -5)$ is a hole for this piece, but the second piece fills it in.

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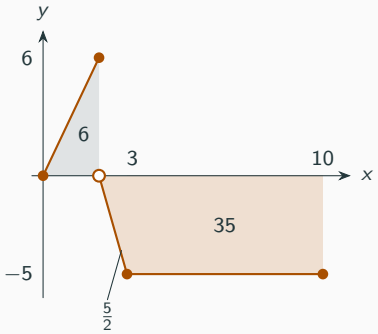
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Example 6, Steps 2–4

Steps 2 and 3:

- above on $[0, 2]$: a triangle, $\frac{1}{2}(2)(6) = 6$
- below on $[2, 3]$: a triangle, $\frac{1}{2}(1)(5) = \frac{5}{2}$
- below on $[3, 10]$: a rectangle, $7(5) = 35$



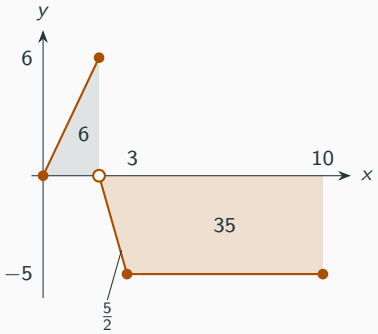
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Step 4:

$$\int_0^{10} g(x)\,dx = 6 - \frac{5}{2} - 35 = -\frac{63}{2}$$



Summary

- Approximate the area under $y = f(x)$ on $[a, b]$ with n rectangles of width $\Delta x = \frac{b-a}{n}$: $A \approx \sum_{i=1}^{n} f(x_i)\Delta x$. More rectangles give a better approximation.
- The definite integral is the limit: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.
- Properties: $\int_a^a f = 0$; constants and sums come out; $\int_a^b f = \int_a^c f + \int_c^b f$; $\int_a^b f = -\int_b^a f$; area above the x-axis counts as positive, area below as negative.
- By geometry: graph, shade, find each area, then add the areas above and subtract the areas below. $\int_0^4 (8 - 2x)\,dx = 16$, $\int_{-4}^2 (2x + 4)\,dx = 12$, $\int_{-5}^3 \big(|x| - 2\big)\,dx = 1$, $\int_0^{10} g(x)\,dx = -\frac{63}{2}$.