Section 7.2: Substitution

Undoing the chain rule to integrate products, quotients and compositions

Example 1: a product that will not split

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Strategy, Step 1: decide

Can the integrand be rewritten as a sum of powers (Section 7.1)? If it is a product, a quotient or a composition that will not split, try a substitution.

Step 2: choose u, the inside function

Look for a function *inside* another function:

- outside: e^x
- inside: x^2 (so e^{x^2} is e^x with x^2 put in for x)

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Strategy, Step 2: choose u

Find the outside function and the inside function, and let u be the inside function. It is often the expression in an exponent, under a root, or inside a power.

Step 3: find du

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Strategy, Step 3: find du

Compute $\frac{du}{dx}$, then multiply both sides by dx: $du = u'(x)\,dx$.

Step 4: rewrite the integral in terms of u

Reorder the factors, then replace x^2 by u and $2x\,dx$ by du:

$$\int 2x\,e^{x^2}\,dx = \int \underbrace{e^{x^2}}_{e^u}\,\underbrace{2x\,dx}_{du} = \int e^u\,du$$

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Strategy, Step 4: substitute

Replace the inside function by u, and the rest of the integrand, together with dx, by du. No x may be left.

Steps 5 and 6: integrate, then substitute back

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Check by the chain rule: $\frac{d}{dx}\left[e^{x^2}\right] = e^{x^2} \cdot 2x$, the integrand. Substitution is the chain rule run backwards.

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Strategy, Steps 5 and 6: integrate, then substitute back

Integrate with respect to u, using the rules from 7.1. Then replace u by the inside function, so the answer is in terms of x. Add $+\,C$, and check by differentiating.

The strategy we just built

Integration by substitution

1. **Decide:** a product, quotient or composition that will not become a sum of powers.
2. **Choose u:** the inside function.
3. **Find du:** $du = u'(x)\,dx$.
4. **Substitute:** rewrite the whole integral in u and du, with no x left.
5. **Integrate** with respect to u.
6. **Substitute back** the inside function, add $+ C$, and check.

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5. **Integrate** with respect to u.
6. **Substitute back** the inside function, add $+ C$, and check.

Step 2 takes practice: a good u is an inside function whose derivative also appears in the integral, up to a constant factor.

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Steps 4–6:

$$\int \underbrace{(x^2 + 2x + 5)^5}_{u^5} \underbrace{(2x + 2)\, dx}_{du} = \int u^5\, du = \frac{u^6}{6} + C = \frac{(x^2 + 2x + 5)^6}{6} + C$$

Example 3: a quotient

$$\int \frac{3x^2}{1+x^3}\,dx$$

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Steps 4–6:

$$\int \underbrace{\left(1+x^3\right)^{-1}}_{u^{-1}} \underbrace{3x^2\,dx}_{du} = \int u^{-1}\,du = \ln|u| + C = \ln\left|1+x^3\right| + C$$

Try it — Example 4

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Steps 1–3: a product with a root. The inside function is under the root: $u = x^3 + 5$, so $du = 3x^2\,dx$.

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Steps 1–3: a product with a root. The inside function is under the root: $u = x^3 + 5$, so $du = 3x^2\,dx$.

Steps 4–6:

$$\int \underbrace{(x^3+5)^{1/2}}_{u^{1/2}}\,\underbrace{3x^2\,dx}_{du} = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}(x^3+5)^{3/2} + C$$

Example 5: when du is off by a constant

$$\int x\, e^{-x^2}\, dx$$

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Step 3: $\frac{du}{dx} = -2x$, so $du = -2x\, dx$. But the integral has $x\, dx$, not $-2x\, dx$, so divide both sides by -2:

$$-\frac{1}{2}\, du = x\, dx$$

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Strategy, Step 3: fixing a constant factor

If du differs from the rest of the integrand by a constant factor, solve for the part that is in the integral. A constant can move outside the integral; a variable cannot.

Example 5, Steps 4–6

$$\int \underbrace{e^{-x^2}}_{e^u} \underbrace{x\,dx}_{-\frac{1}{2}\,du} = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}\,e^u + C = -\frac{1}{2}\,e^{-x^2} + C$$

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Check: $\frac{d}{dx}\left[-\frac{1}{2}\,e^{-x^2}\right] = -\frac{1}{2}\,e^{-x^2}\cdot(-2x) = x\,e^{-x^2}$, the integrand.

Example 6

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Steps 1–3: a quotient, rewritten as a power. $u = 4x + 7$, $\frac{du}{dx} = 4$, so $du = 4\,dx$ and $\frac{1}{4}\,du = dx$.

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Steps 4–6:

$$\int \underbrace{(4x+7)^{-1}}_{u^{-1}} \underbrace{dx}_{\frac{1}{4}\,du} = \frac{1}{4}\int u^{-1}\,du = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|4x+7| + C$$

Try it — Example 7

$$\int \frac{x^3}{\sqrt{9-x^4}}\,dx = \int x^3\left(9-x^4\right)^{-1/2}\,dx$$

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Steps 1–3: $u = 9 - x^4$, $\frac{du}{dx} = -4x^3$, so $du = -4x^3\,dx$ and $-\frac{1}{4}\,du = x^3\,dx$.

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Steps 4–6:

$$\int \underbrace{\left(9-x^4\right)^{-1/2}}_{u^{-1/2}} \underbrace{x^3\,dx}_{-\frac{1}{4}\,du} = -\frac{1}{4}\int u^{-1/2}\,du = -\frac{1}{4}\cdot\frac{u^{1/2}}{1/2} + C = -\frac{1}{2}\left(9-x^4\right)^{1/2} + C$$

Example 8: a composition

$$\int \sin(7x)\, dx$$

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Steps 1–3: the inside function is $7x$: $u = 7x$, $du = 7\,dx$, so $\frac{1}{7}\,du = dx$.

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Steps 4–6: since $\frac{d}{du}\big[-\cos(u)\big] = \sin(u)$,

$$\int \sin(\underbrace{7x}_{u})\,\underbrace{dx}_{\frac{1}{7}\,du} = \frac{1}{7}\int \sin(u)\,du = -\frac{1}{7}\cos(u) + C = -\frac{1}{7}\cos(7x) + C$$

Example 9

$$\int \sin^2(x)\cos(x)\,dx, \qquad \text{where } \sin^2(x) = \big(\sin(x)\big)^2$$

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Steps 1–3: a product. The square is the outside function, so $u = \sin(x)$, $\frac{du}{dx} = \cos(x)$, and $du = \cos(x)\,dx$.

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Steps 4–6:

$$\int \underbrace{\big(\sin(x)\big)^2}_{u^2}\,\underbrace{\cos(x)\,dx}_{du} = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3(x)}{3} + C$$

Example 10

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Steps 1–3: a quotient, rewritten as a product. $u = \cos(x)$, $du = -\sin(x)\,dx$, so $-du = \sin(x)\,dx$.

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Steps 4–6:

$$\int \underbrace{\big(\cos(x)\big)^{-1/2}}_{u^{-1/2}}\,\underbrace{\sin(x)\,dx}_{-du} = -\int u^{-1/2}\,du = -\frac{u^{1/2}}{1/2} + C = -2\sqrt{\cos(x)} + C$$

Try it — Example 11

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \sin(x)\big(\cos(x)\big)^{-1}\,dx$$

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$$\int \underbrace{\big(\cos(x)\big)^{-1}}_{u^{-1}}\,\underbrace{\sin(x)\,dx}_{-du} = -\int u^{-1}\,du = -\ln|u| + C = -\ln\big|\cos(x)\big| + C$$

Summary

Substitution strategy

(1) decide that the integrand will not become a sum of powers; (2) choose u, the inside function; (3) find $du = u'(x)\,dx$, fixing a constant factor if needed; (4) rewrite the integral in u and du; (5) integrate; (6) substitute back, add $+\,C$, and check.

- Inside an exponent: $\int 2x\,e^{x^2}\,dx = e^{x^2} + C$; $\int x\,e^{-x^2}\,dx = -\frac{1}{2}\,e^{-x^2} + C$
- Inside a power or root: $\int 3x^2\sqrt{x^3+5}\,dx = \frac{2}{3}\left(x^3+5\right)^{3/2} + C$
- A power of -1 gives a logarithm: $\int \frac{3x^2}{1+x^3}\,dx = \ln\left|1+x^3\right| + C$
- Trigonometric: $\int \sin(7x)\,dx = -\frac{1}{7}\cos(7x) + C$; $\int \tan(x)\,dx = -\ln\left|\cos(x)\right| + C$