Section 7.1: Antiderivatives

Undoing the derivative, and finding a function from its rate of change

Antiderivatives

So far we have gone from a function to its derivative. Now we go backwards: from a rate of change back to the function.

Definition

F is an **antiderivative** of f on (a, b) if $F'(x) = f(x)$ for every x in (a, b).

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Example. Let $f(x) = \cos x$ on $(-\infty, \infty)$. Then $F(x) = \sin x$ is an antiderivative of f, because

$$\frac{d}{dx}\left[\sin x\right] = \cos x.$$

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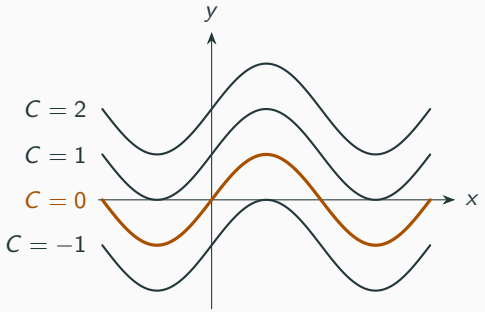
Any others? $\sin x + 1$, $\sin x - 5$, $\sin x + 100$, ... The derivative of a constant is 0, so each of these is an antiderivative of $\cos x$ too.

The most general antiderivative

All the antiderivatives of $\cos x$

$$F(x) = \sin x + C,$$

where C is any real number. This is the **most general antiderivative** of $\cos x$.



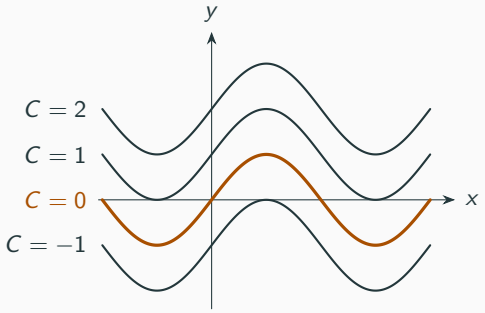
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The graphs of $\sin x + C$ are the same curve shifted up or down. At every x they have the same slope, $\cos x$.



Notation: the indefinite integral

$$\int f(x)\,dx = F(x) + C$$

- $\int$ is the **integral sign**
- $f(x)$ is the **integrand**, the function we are antidifferentiating
- dx says we are finding the antiderivative **with respect to** x
- $F(x) + C$ is the **most general antiderivative** of f with respect to x

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Example

$$\int \cos x\,dx = \sin x + C$$

Antiderivatives of powers of x: find the pattern

Find the most general antiderivative:

- $\int x\,dx =$
- $\int x^2\,dx =$
- $\int x^3\,dx =$
- $\int x^4\,dx =$

Antiderivatives of powers of x: find the pattern

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The Reverse Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

Add 1 to the exponent, then divide by the new exponent.

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Check by differentiating: $\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = \frac{(n+1)\,x^n}{n+1} = x^n$.

The exception $n = -1$, and rewriting first

The exception. For $n = -1$ the Reverse Power Rule would divide by $n + 1 = 0$. Instead, since $\frac{d}{dx} \ln|x| = \frac{1}{x}$,

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C$$

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Rewrite as a power first:

$$\int \frac{1}{x^3}\, dx = \int x^{-3}\, dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

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$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

Properties of indefinite integrals

Constant multiple rule

$$\int k\, f(x)\, dx = k \int f(x)\, dx \quad \text{for any real number } k$$

Sum and difference rule

$$\int \big[f(x) \pm g(x)\big]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

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These let us split an integral into pieces and pull constants out. There is **no** simple product rule or quotient rule for integrals, so a product or a quotient has to be rewritten as a sum first.

Example 1: building a strategy

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Antiderivative strategy, Step 1: rewrite

Write each term as a constant times a power of x: roots become fractional powers, $\frac{1}{x^n}$ becomes x^{-n}, and products, quotients and powers of sums are multiplied out or split into separate terms.

Step 2: split into pieces

Split at each $+$ and $-$ (sum and difference rule), and pull out the constants (constant multiple rule):

$$\int \left(3x^4 - 5x^{2/3} - x^{-1}\right) dx = 3\int x^4\, dx - 5\int x^{2/3}\, dx - \int x^{-1}\, dx$$

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Antiderivative strategy, Step 2: split

Split the integral at each $+$ and $-$, and pull each constant out in front of its integral.

Step 3: antidifferentiate each piece

$$3\int x^4\,dx - 5\int x^{2/3}\,dx - \int x^{-1}\,dx = 3\cdot\frac{x^5}{5} - 5\cdot\frac{x^{5/3}}{5/3} - \ln|x| + C$$

- x^4 and $x^{2/3}$: the Reverse Power Rule, with $4+1=5$ and $\frac{2}{3}+1=\frac{5}{3}$
- x^{-1}: the exception, $\ln|x|$

Step 3: antidifferentiate each piece

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Antiderivative strategy, Step 3: use the rules

Reverse Power Rule for x^n with $n \neq -1$, and $\ln|x|$ for x^{-1}. Add one $+C$ at the end.

Step 4: simplify, then check

Dividing by $\frac{5}{3}$ is the same as multiplying by $\frac{3}{5}$, so $5 \cdot \dfrac{x^{5/3}}{5/3} = 3x^{5/3}$:

$$\int \left(3x^4 - 5x^{2/3} - x^{-1}\right) dx = \frac{3}{5}x^5 - 3x^{5/3} - \ln|x| + C$$

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Check: $\dfrac{d}{dx}\left[\dfrac{3}{5}x^5 - 3x^{5/3} - \ln|x| + C\right] = 3x^4 - 5x^{2/3} - \dfrac{1}{x}$, the integrand.

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Antiderivative strategy, Step 4: simplify and check

Simplify the coefficients. Then differentiate your answer: you should get the integrand back.

The strategy we just built

Finding an indefinite integral

1. **Rewrite** each term as a constant times a power of x.
2. **Split** at each $+$ and $-$, and pull out the constants.
3. **Use the rules** on each piece: Reverse Power Rule, or $\ln|x|$ for x^{-1}. Add one $+ C$.
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Step 1 is where the thinking happens. A root, a fraction, a product or a power of a sum must become a sum of powers before the rules apply.

Try it — Example 2

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Step 4 (simplify): $-3x^{-1} - 2\sqrt{2}\,x^{1/2} + 27x + C \;=\; -\dfrac{3}{x} - 2\sqrt{2x} + 27x + C$

Example 3: a quotient — split it into a sum

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$$\frac{x^3}{x^3} + \frac{\sqrt{2}\, x^{1/2}}{x^3} + \frac{x}{x^3} = 1 + \sqrt{2}\, x^{-5/2} + x^{-2}$$

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Steps 2 and 3: $\displaystyle\int 1\, dx + \sqrt{2} \int x^{-5/2}\, dx + \int x^{-2}\, dx = x + \sqrt{2} \cdot \frac{x^{-3/2}}{-3/2} + \frac{x^{-1}}{-1} + C$

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Step 4 (simplify): $\displaystyle x - \frac{2\sqrt{2}}{3}\, x^{-3/2} - x^{-1} + C$

Try it — Example 4: a power of a sum

$$\int \left(x^2 - 4\right)^2 \, dx$$

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Step 1 (rewrite): there is no rule for a power of a sum, so multiply it out:

$$(x^2 - 4)^2 = (x^2 - 4)(x^2 - 4) = x^4 - 8x^2 + 16$$

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Steps 2–4: $\displaystyle\int (x^4 - 8x^2 + 16)\, dx = \frac{x^5}{5} - \frac{8x^3}{3} + 16x + C$

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Watch out: $\frac{1}{3}(x^2 - 4)^3$ is **not** an antiderivative. By the chain rule, its derivative is $2x(x^2 - 4)^2$.

Example 5: a growing population of bacteria

Example 5. Suppose that a population of bacteria grows at a rate that slows down with time, so that the growth rate after t days is

$$P'(t) = \frac{5000}{\sqrt{t}} \qquad \text{for } t \geq 1.$$

Suppose that when $t = 1$, the population is 25,000.

(a) Find the function $P(t)$ giving the population at time t.

(b) How many days will it take for the population to reach 115,000?

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We know the rate $P'(t)$ and one value, $P(1)$: this is called an **initial value problem**. We solve it step by step and build a second strategy.

Example 5, Step 1: antidifferentiate

$P'(t) = \dfrac{5000}{\sqrt{t}}$ for $t \geq 1$, and $P(1) = 25{,}000$. (a) Find $P(t)$.

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Step 1: antidifferentiate. The variable is t, so we integrate with respect to t:

$$P(t) = \int 5000\, t^{-1/2}\, dt = 5000 \cdot \frac{t^{1/2}}{1/2} + C = 10{,}000\, t^{1/2} + C$$

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Every value of C gives a population growing at the rate $\frac{5000}{\sqrt{t}}$. Which one is ours?

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Initial value problem, Step 1: antidifferentiate

The rate f' gives the function up to a constant: $f(x) + C$.

Example 5, Steps 2 and 3: find C, then write the function

We know $P(1) = 25{,}000$. Put $t = 1$ into $P(t) = 10{,}000\, t^{1/2} + C$:

$$25{,}000 = 10{,}000\,(1)^{1/2} + C \qquad \Longrightarrow \qquad C = 15{,}000$$

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Answer to (a)

$P(t) = 10{,}000\,\sqrt{t} + 15{,}000$

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Initial value problem, Steps 2 and 3: find C, write the function

Put the known point $\big(a, f(a)\big)$ into $f(x) + C$ and solve for C. Then write f with that value of C.

Example 5, Step 4: answer the question

(b) How many days until the population reaches 115,000? Solve $P(t) = 115{,}000$:

$$115{,}000 = 10{,}000\sqrt{t} + 15{,}000 \implies 100{,}000 = 10{,}000\sqrt{t} \implies \sqrt{t} = 10 \implies t = 100$$

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Answer to (b)

The population reaches 115,000 after **100 days**.

Check: $P(100) = 10{,}000\,(10) + 15{,}000 = 115{,}000$.

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$$115{,}000 = 10{,}000\sqrt{t} + 15{,}000 \implies 100{,}000 = 10{,}000\sqrt{t} \implies \sqrt{t} = 10 \implies t = 100$$

Answer to (b)

The population reaches 115,000 after **100 days**.

Check: $P(100) = 10{,}000\,(10) + 15{,}000 = 115{,}000$.

Initial value problem, Step 4: answer

Use the function to answer the question that was asked, with units.

Initial value problems: the strategy we just built

In an **initial value problem** we know a rate of change f' and one value $f(a)$, and we want f. The two pieces of information do two different jobs: f' gives $f(x) + C$, and the point $\big(a, f(a)\big)$ gives C.

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Solving an initial value problem

1. **Antidifferentiate** the rate to get $f(x) + C$.
2. **Find** C by putting in the known value $f(a)$.
3. **Write the function** with that value of C.
4. **Answer the question**, with units.

One more rule: exponentials

Warm-up: find $\int e^{2x}\,dx$.

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Exponential rule

$$\int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + C, \qquad k \neq 0$$

In particular, $\int e^{x}\,dx = e^{x} + C$.

Example 6: excretion of a compound

Example 6. The rate of excretion of a biochemical compound is $f'(t) = 0.01\, e^{-0.01t}$, and the total amount excreted by time t (in minutes) is $f(t)$.

(a) Find an expression for $f(t)$. (b) If 0 units are excreted at time $t = 0$, how many units are excreted in 10 minutes?

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Step 4: answer

$f(10) = 1 - e^{-0.1} \approx 0.0952$: about 0.0952 units are excreted in the first 10 minutes.

Example 7: blood volume in the aorta

Example 7. The volume $V(t)$ of blood in the aorta at time t changes at the rate

$$V'(t) = -kP(t),$$

where $P(t)$ is the pressure in the aorta at time t, and k is a constant that depends on properties of the aorta. (The rate at which the blood volume changes is proportional to the pressure.) The pressure is

$$P(t) = P_0\, e^{-mt},$$

where P_0 is the pressure at time $t = 0$ and m is another constant. Letting V_0 be the volume at $t = 0$, find a formula for $V(t)$.

Example 7, Steps 1–3 and a follow-up

Step 1: $V(t) = \int -kP_0\, e^{-mt}\, dt = -kP_0 \cdot \dfrac{e^{-mt}}{-m} + C = \dfrac{kP_0}{m}\, e^{-mt} + C$

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Step 2: $V(0) = V_0$ gives $V_0 = \dfrac{kP_0}{m}\, e^0 + C = \dfrac{kP_0}{m} + C$, so $C = V_0 - \dfrac{kP_0}{m}$

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Step 3: the volume of blood in the aorta at time t

$$V(t) = \frac{kP_0}{m}\, e^{-mt} + V_0 - \frac{kP_0}{m}$$

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Follow-up: what happens as $t \to \infty$? The pressure decreases, so $m > 0$ and $e^{-mt} \to 0$: $V(t) \to V_0 - \dfrac{kP_0}{m}$. The volume approaches this final steady value.

Summary

- F is an antiderivative of f if $F' = f$. The most general antiderivative is written $\int f(x)\,dx = F(x) + C$.
- Rules: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$; $\int x^{-1}\,dx = \ln|x| + C$; $\int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + C$; constants and sums come out of the integral.
- **Indefinite integral strategy:** rewrite as powers, split, use the rules (one $+\,C$), simplify and check.
- **Initial value problem strategy:** antidifferentiate, find C from the known value, write the function, answer.
- Bacteria: $P(t) = 10{,}000\sqrt{t} + 15{,}000$ reaches 115,000 after 100 days. Excretion: $f(10) = 1 - e^{-0.1} \approx 0.0952$ units. Aorta: $V(t)$ approaches $V_0 - \frac{kP_0}{m}$.

Next time: skill practice with antiderivatives.