Section 6.4: Related Rates

How fast one quantity changes when it is linked to another changing quantity

Example 1: a thermal camera watching seals

In a **related rates** problem, quantities that change with time are linked by an equation, so their rates of change are linked too. We use that link to find one rate from another.

Example 1. A thermal camera is mounted on a tower 500 m offshore to monitor seal activity along the beach at night. The camera slowly rotates at a constant 1.5 radians/min, sweeping its infrared beam across the shoreline.

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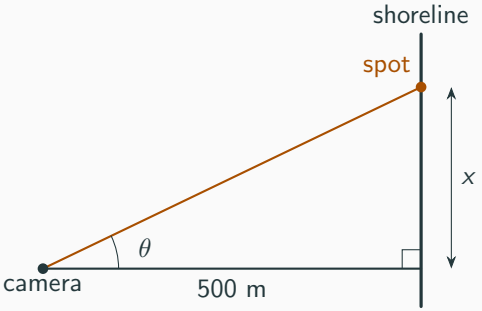
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We solve it one step at a time. After each step, a **Strategy** box says what that step is in general. Together the boxes make a strategy for every related rates problem.

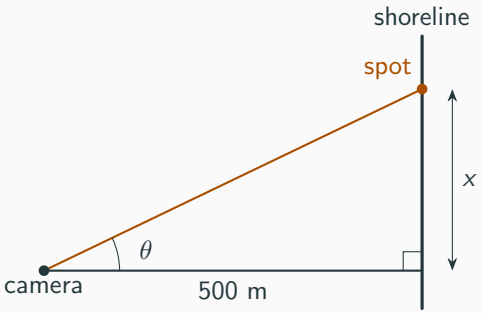
Step 1: draw a picture and label it

- $\theta =$ the angle of the beam, measured from the line to the nearest point of the shore
- $x =$ the distance along the shoreline from that point to the scanned spot
- the 500 m from the camera to the shore never changes



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Strategy, Step 1: picture

Draw the situation. Label every quantity, using letters for the ones that change.

Steps 2 and 3: what changes, what is given, what is asked

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- **Task:** find $x'(t)$, that is, $\frac{dx}{dt}$

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Strategy, Steps 2 and 3: sort the quantities

List the **functions of time** (they change) and the **constants** (they don't). Write down the **given** values and rates, with units, and the **task**: the rate you want.

Step 4: an equation that is true at every moment

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Strategy, Step 4: the equation

Find an equation that links the changing quantities: a right triangle (trigonometry or the Pythagorean theorem), similar triangles, or a formula given in the problem. It must be true at every moment.

Step 5: differentiate both sides with respect to t

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Strategy, Step 5: differentiate

Differentiate both sides with respect to t. Every function of t brings its derivative along (chain rule), and constants have derivative 0.

Step 6: substitute the values at that moment, then solve

Now use the moment we care about. We need $\sec^2\theta$, and $\sec^2\theta = 1 + \tan^2\theta$ with $\tan\theta = \dfrac{400}{500}$:

$$\left(1 + \left(\frac{400}{500}\right)^2\right)(1.5) = \frac{x'(t)}{500}$$

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Strategy, Step 6: substitute, then solve

Only *after* differentiating, put in the values at the moment of interest and solve for the unknown rate.

Step 7: answer the question

Answer

The scanned spot is moving along the shoreline at **1230 m/min** when it is 400 m down the shore.

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Strategy, Step 7: answer

Answer in a sentence, with units. A positive rate means the quantity is increasing; a negative rate means it is decreasing.

The strategy we just built

Solving a related rates problem

1. **Picture:** draw it, with letters for the changing quantities.
2. **Functions of time and constants:** which quantities change and which don't.
3. **Given and task:** the values and rates you know, with units, and the rate you want.
4. **Equation:** it links the changing quantities and is true at every moment.
5. **Differentiate both sides with respect to** t (chain rule).
6. **Substitute the values at that moment, then solve** for the unknown rate.
7. **Answer the question** in a sentence, with units.

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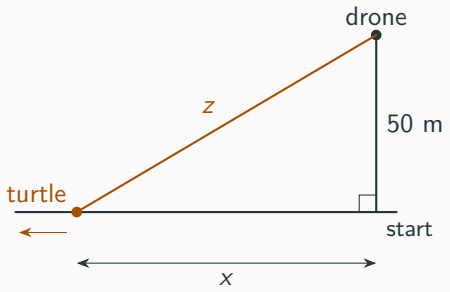
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The most common slip: putting in the moment's values *before* differentiating.

Try it — Example 2: a drone and a sea turtle

Example 2. Marine biologists are using a drone flying 50 m above the water to monitor a sea turtle swimming parallel to the shore. The turtle moves at a steady 1 m/s, and the drone hovers above a point on the beach directly inland from the turtle's starting position.

How fast is the distance between the drone and the turtle increasing when the turtle is 30 m from the point directly below the drone?



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Step 4 (equation): the Pythagorean theorem in the right triangle, true at every moment:

$$\big(x(t)\big)^2 + 50^2 = \big(z(t)\big)^2$$

Example 2, Steps 5–7

Step 5 (differentiate):

$$2\,x(t)\,x'(t) + 0 = 2\,z(t)\,z'(t) \qquad \Longrightarrow \qquad z'(t) = \frac{x(t)\,x'(t)}{z(t)}$$

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Step 6 (substitute and solve): at that moment the equation from Step 4 gives $z = \sqrt{30^2 + 50^2} = \sqrt{3400} \approx 58.3$ m, so

$$z'(t) = \frac{30\,(1)}{\sqrt{30^2 + 50^2}} \approx 0.51$$

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Step 7: answer

The distance between the drone and the turtle is increasing at $\frac{30}{\sqrt{30^2+50^2}} \approx 0.51$ m/s.

Example 3: a rising parcel of air

Example 3. Consider a parcel of air rising quickly in the atmosphere. As the parcel rises, it expands and its temperature changes. If the parcel does not exchange heat with the surrounding air, its volume V and its temperature T are related by

$$T\,V^{\gamma-1} = C,$$

where $\gamma \approx 1.4$ for sufficiently dry air and C is a constant. The temperature is measured in kelvin, a scale chosen so that the temperature is always positive. Rising air expands, so the volume V of the parcel increases with time t.

How does the rate of change of the temperature, $\frac{dT}{dt}$, depend on the rate of change of the volume, $\frac{dV}{dt}$? Determine how the temperature of the air parcel changes as it rises.

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Step 5 (differentiate): the product rule on the left, with the chain rule on $\left(V(t)\right)^{\gamma-1}$:

$$\frac{dT}{dt}\,V^{\gamma-1} + T\,(\gamma-1)\,V^{\gamma-2}\,\frac{dV}{dt} = 0$$

Example 3, Steps 6 and 7: solve and interpret

Step 6 (solve for $\frac{dT}{dt}$): move the second term across and divide by $V^{\gamma-1}$:

$$\frac{dT}{dt} = \frac{-T\,(\gamma-1)\,V^{\gamma-2}\,\frac{dV}{dt}}{V^{\gamma-1}} = -(\gamma-1)\,\frac{T}{V}\,\frac{dV}{dt}$$

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Step 7: interpret

T and V are positive. If the air expands ($\frac{dV}{dt} > 0$), then $\frac{dT}{dt} < 0$: the temperature **decreases**. So a rising parcel of air cools, and a falling parcel warms.

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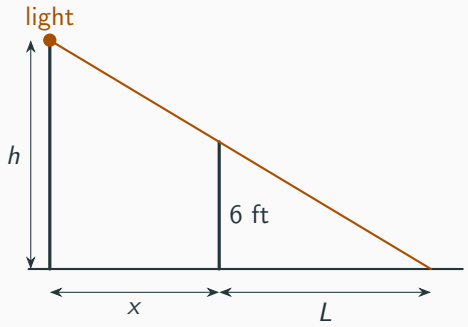
T and V are positive. If the air expands ($\frac{dV}{dt} > 0$), then $\frac{dT}{dt} < 0$: the temperature **decreases**. So a rising parcel of air cools, and a falling parcel warms.

There were no values to put in at Step 6, so the answer is a general law, and its sign tells the story.

If time — Example 4: a researcher's shadow

Example 4. A 6-ft-tall wildlife researcher is walking away from a stationary observation light at 50 ft/min while monitoring nocturnal animals. The light casts the researcher's shadow along the ground. When the researcher is 8 ft from the light, the shadow is 10 ft long.

How fast is the length of the shadow increasing when the researcher is 25 ft from the light?



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Constants: the researcher's height, 6 ft, and the height h of the light. Find h from the snapshot "8 ft from the light, shadow 10 ft" with similar triangles:

$$\frac{h}{8+10} = \frac{6}{10} \implies 10h = 18(6) \implies h = 10.8 \text{ ft}$$

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Given: $x'(t) = 50$ ft/min and $x(t) = 25$ ft (the moment we care about). **Task:** $L'(t)$.

Step 4 (equation): similar triangles, at every moment:

$$\frac{10.8}{x(t)+L(t)} = \frac{6}{L(t)} \implies 6\,x(t) + 6\,L(t) = 10.8\,L(t)$$

Example 4, Steps 5–7

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Step 7: answer

The researcher's shadow is getting longer at **62.5 ft/min** when the researcher is 25 ft from the light.

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Step 7: answer

The researcher's shadow is getting longer at **62.5 ft/min** when the researcher is 25 ft from the light.

Notice that $x = 25$ was never used: $6x = 4.8L$ means $L = 1.25x$ at every moment, so $L' = 1.25(50)$ wherever the researcher is.

Summary

Related rates strategy

(1) picture; (2) functions of time and constants; (3) given values and rates, and the task; (4) an equation that is true at every moment; (5) differentiate both sides with respect to t; (6) substitute the values at that moment, then solve; (7) answer with units.

- Camera and seals: $\tan\theta = \frac{x}{500}$; the spot moves at 1230 m/min when it is 400 m down the shore.
- Drone and turtle: $x^2 + 50^2 = z^2$; the distance grows at $\frac{30}{\sqrt{3400}} \approx 0.51$ m/s.
- Air parcel: $T\,V^{\gamma-1} = C$ gives $\frac{dT}{dt} = -0.4\,\frac{T}{V}\,\frac{dV}{dt}$, so rising air cools.
- Shadow: $\frac{10.8}{x+L} = \frac{6}{L}$; the shadow grows at 62.5 ft/min.
- Tips: substitute only after differentiating; a value you are missing at that moment comes from the Step 4 equation.