Section 6.2: Applications of Extrema (Part 2)

Optimization on a closed interval: a postal package and a homing pigeon

Recap: the optimization strategy

The same steps, every time

1. **Picture and variables**, with units.
2. **Objective function:** the quantity to make as large or as small as possible.
3. **Constraint equation(s):** what is fixed.
4. **Objective in terms of one variable:** use the constraints to remove the other variables.
5. **Interval of interest:** the values that make sense.
6. **Absolute max or min** on that interval: derivative, critical points, check.
7. **Answer the question**, with units.

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Today the interval of interest is closed, and there are two ways to do the check in Step 6.

Strategy: two ways to check the absolute max or min

Option 1: one critical point (any interval)

If there is **exactly one** critical point in the interval:

- classify it with the **second derivative test** (or a sign chart for the first derivative);
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Option 2: the EVT table (closed interval)

If the function is continuous on a closed, bounded interval $[a, b]$, the EVT applies:

- evaluate it at the critical points **and** at both endpoints;
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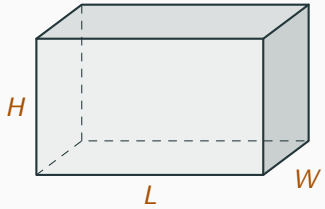
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On a closed interval either option works. Option 1 also works on open intervals such as $(0, \infty)$.

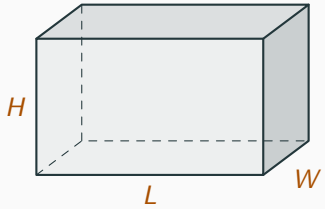
Example 1: the largest package

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Step 1 (picture and variables):
the length L, width W, and height H of the package, in inches.

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Two constraint equations remove two variables, so everything is written in terms of W.

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A **closed, bounded** interval this time. Find the value of W that makes $V(W) = 1350W^2 - 30W^3$ **as big as possible** on $[0, 45]$.

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Strategy tip: each dimension gives an inequality; together they give the interval of interest.

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The only critical point

$W = 30$ is the only critical point of $V(W)$ in $[0, 45]$.

Example 1: does $W = 30$ give the absolute max?

Option 1: second derivative test

$$V''(W) = 2700 - 180W$$

$V''(30) = 2700 - 180(30) = -2700 < 0$

Concave down, so a local max.
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Step 7: answer

$W = 30$ in, $L = 5(30) = 150$ in, $H = 270 - 6(30) = 90$ in (volume 405,000 in^3)

Example 2: the homing pigeon

Example. Homing pigeons avoid flying over large bodies of water, preferring to fly around them. Assume that a pigeon released from a boat 1 mile from the shore of a lake (point B) flies first to point P on the shore and then along the straight edge of the lake to reach its home at L. If L is 2 miles from point A, the point on the shore closest to the boat, and if a pigeon needs $\frac{4}{3}$ as much energy per mile to fly over water as over land, find the location of point P which minimizes the energy used.

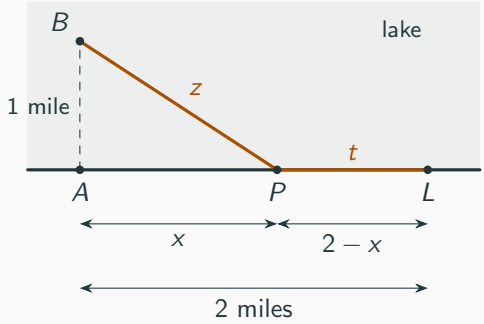
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Reading the problem

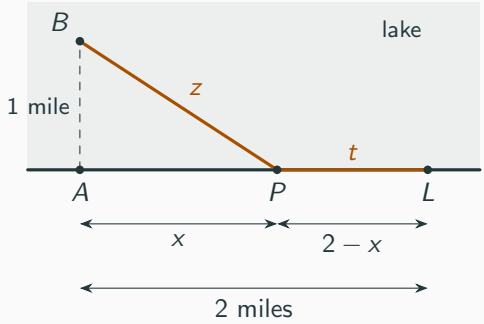
The objective is the **energy**. The rest fixes the picture: the boat is 1 mile out, and home is 2 miles along the shore from A. The unknown is where P is.

Example 2, Steps 1 and 3: picture and constraints



Let x = the distance from A to P, in miles.

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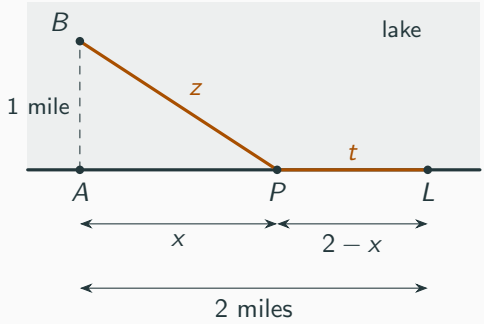


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$$1^2 + x^2 = z^2 \implies z = \sqrt{1 + x^2}$$

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(2) Over land, from P to L:

$$t = 2 - x$$

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Energy = (energy rate) $\times$ (distance). Use a rate of 1 energy unit per mile over land; then the rate over water is $\frac{4}{3}$.

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Find the value of x that makes $E(x) = \frac{4}{3}\sqrt{1+x^2} + 2 - x$ **as small as possible** on $[0, 2]$.

Example 2, Step 6: the critical points

Find $E'(x)$ (chain rule on the square root):

$$E'(x) = \frac{4}{3} \cdot \frac{1}{2} \left(1 + x^2\right)^{-1/2} \cdot 2x - 1 = \frac{4x}{3\sqrt{1 + x^2}} - 1$$

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Where $E'(x) = 0$**:**

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Strategy tip: squaring both sides can create extra solutions, so check each one in the equation from before you squared.

Example 2: does $x = \sqrt{9/7}$ give the absolute min?

Option 1: the EVT table

E is continuous on $[0, 2]$, so the EVT applies.

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Option 2: second derivative test

$$E''(x) = \frac{4}{3\left(1 + x^2\right)^{3/2}} > 0$$

for every x: concave up, so $x = \sqrt{9/7}$ gives a local min. It is the only critical point on $[0, 2]$, so it gives the absolute min.

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(A sign chart for $E'(x)$, the first derivative test, works here too.)

Example 2, Step 7: answer

Answer

The pigeon should reach the shore at the point P that is $\sqrt{\frac{9}{7}} = \frac{3}{\sqrt{7}} \approx 1.13$ miles from A (about 0.87 miles from home).

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$$E\left(\sqrt{\tfrac{9}{7}}\right) = \frac{4}{3} \cdot \frac{4}{\sqrt{7}} + 2 - \frac{3}{\sqrt{7}} = 2 + \frac{16 - 9}{3\sqrt{7}} = 2 + \frac{\sqrt{7}}{3} \approx 2.88,$$

compared with $\frac{10}{3} \approx 3.33$ for flying straight to A and $\frac{4}{3}\sqrt{5} \approx 2.98$ for flying straight home.

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compared with $\frac{10}{3} \approx 3.33$ for flying straight to A and $\frac{4}{3}\sqrt{5} \approx 2.98$ for flying straight home. The best route is a little *longer* than flying straight home, but it crosses less water.

Summary

Optimization strategy

(1) picture and variables; (2) objective function; (3) constraint equation(s); (4) objective in one variable; (5) interval of interest; (6) absolute max or min on that interval; (7) answer the question, with units.

Two ways to check the absolute max or min

Option 1: exactly one critical point, classified by the second derivative test (or a sign chart). Option 2, on a closed interval: the EVT table of critical points and endpoints.

- Package: maximize $V(W) = 5W^2(270 - 6W)$ on $[0, 45]$: $W = 30$ in, $L = 150$ in, $H = 90$ in.
- Pigeon: minimize $E(x) = \frac{4}{3}\sqrt{1 + x^2} + 2 - x$ on $[0, 2]$: P is $\sqrt{9/7} \approx 1.13$ miles from A.
- Tips: each dimension ≥ 0 gives the interval of interest; after squaring an equation, check the solutions.