## Section 6.2: Applications of Extrema (Part 1)

Optimization: setting up and solving maximum and minimum problems

# Calculus can solve fundamental problems in biology

**Maximum sustainable harvest (a life sciences application)**

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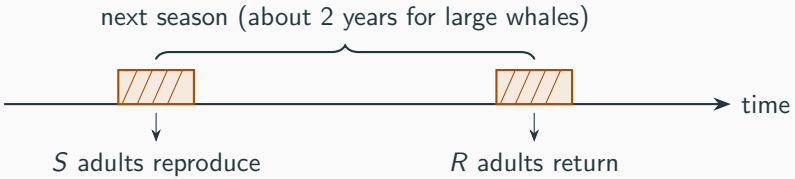
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**Note:** $f(S)$ is hard to describe, because accurate counts are difficult to obtain and many hypotheses can be made about life stages.

**Definition**

Let $S_0$ be the number of spawners that allows as large a harvest as possible without threatening the population with extinction. Then $H(S_0)$ is called the maximum sustainable harvest.

# Example: Idaho rabbits

**Example.** Suppose the spawner-recruit function for Idaho rabbits is

$$f(S) = 2.17\sqrt{S}\,\ln(S+1),$$

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**Plan for today**

For the purpose of this lecture, we will solve **simpler problems** first in order to develop the skill, with one strategy for every problem. (We come back to the rabbits at the end.)

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Step 6 is last lecture. The new skill is Steps 1–5: turning words into a function of one variable.

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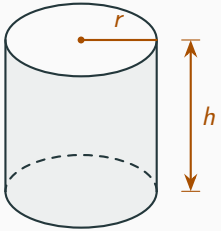
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**Confirming the max or min**

With **exactly one** critical point in the interval, a local min (max) there is the absolute min (max). The second derivative test classifies it quickly.

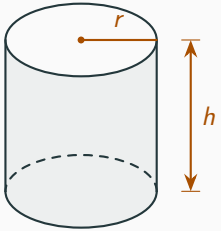
# Example 1: a cylindrical cancer cell

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**Step 1 (picture and variables):**
$r$ = the radius and $h$ = the height of the cell, both in micrometers ($\mu$m).

## Example 1, Steps 2–5: set up the function

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**Step 4 (objective in terms of $r$):** solve the constraint for $h$ and substitute:

$$h = \frac{65}{\pi r^2} \qquad \Longrightarrow \qquad S(r) = 2\pi r^2 + 2\pi r \left(\frac{65}{\pi r^2}\right) = 2\pi r^2 + \frac{130}{r}$$

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**The only critical point**

$r = \sqrt[3]{65/(2\pi)} \approx 2.18$ is the only critical point of $S(r)$ on $(0, \infty)$.

## Example 1: does $r = \sqrt[3]{65/(2\pi)}$ give the absolute min?

**Second derivative test:** $S''(r) = 4\pi + \dfrac{260}{r^3}$. At the critical point $r^3 = \frac{65}{2\pi}$, so

$$S''\left(\sqrt[3]{\tfrac{65}{2\pi}}\right) = 4\pi + \frac{260}{65/(2\pi)} = 4\pi + 8\pi = 12\pi > 0 \qquad \text{(concave up: a local min)}$$

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**Step 7: answer**

$$r = \sqrt[3]{\frac{65}{2\pi}} \approx 2.18\,\mu\text{m} \qquad \text{and} \qquad h = \frac{65}{\pi\left(\sqrt[3]{65/(2\pi)}\right)^2} \approx 4.36\,\mu\text{m}$$

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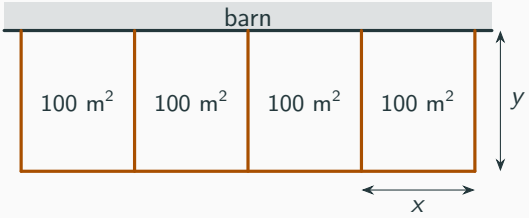
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A pattern: $h = 2r$. The cell with the least surface area is exactly as tall as it is wide.

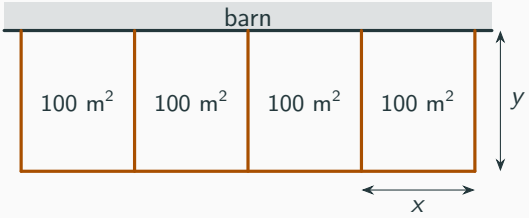
# Try it — Example 2: fencing four pens

**Example.** A rancher plans to make 4 identical and adjacent rectangular pens against a barn, each with an area of 100 $m^2$. Find the dimensions of each pen that minimize the amount of fence used.



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No fence is needed along the barn: there are 5 fences of length $y$ and one front fence of length $4x$.

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Only $x = \sqrt{125} \approx 11.18$ is in $(0, \infty)$.

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**The only critical point**

$x = \sqrt{125}$ is the only critical point of $P(x)$ on $(0, \infty)$.

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**Second derivative test:** $P''(x) = \dfrac{1000}{x^3}$, so

$$P''\left(\sqrt{125}\right) = \frac{1000}{\left(\sqrt{125}\right)^3} > 0 \qquad \text{(concave up: a local min)}$$

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Each pen is $x = \sqrt{125} = 5\sqrt{5} \approx 11.18$ m wide and $y = \dfrac{100}{\sqrt{125}} = 4\sqrt{5} \approx 8.94$ m deep.

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Each pen is $x = \sqrt{125} = 5\sqrt{5} \approx 11.18$ m wide and $y = \dfrac{100}{\sqrt{125}} = 4\sqrt{5} \approx 8.94$ m deep.

The fence used is then $P\left(\sqrt{125}\right) = 40\sqrt{5} \approx 89.4$ m, the least possible.

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- **Critical points:** $H'(S) = f'(S) - 1 = 0$, that is, $f'(S) = 1$.

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- **Critical points:** $H'(S) = f'(S) - 1 = 0$, that is, $f'(S) = 1$.
- This equation cannot be solved with algebra alone: use a **graphing calculator** to find $S_0$, then check that it gives the absolute max, just as in the examples.

# Summary

## Optimization strategy

(1) picture and variables; (2) objective function; (3) constraint equation; (4) objective in one variable; (5) interval of interest; (6) absolute max or min on that interval; (7) answer the question, with units.

## Confirming the absolute max or min

Exactly one critical point in the interval, and a local min (max) there $\Rightarrow$ the absolute min (max). The second derivative test is a quick way to classify the critical point.

- Harvest: maximize $H(S) = f(S) - S$; $H(S_0)$ is the maximum sustainable harvest.
- Cylindrical cell with volume $65\,\mu\text{m}^3$: $r = \sqrt[3]{65/(2\pi)} \approx 2.18\,\mu\text{m}$ and $h = 2r \approx 4.36\,\mu\text{m}$.
- Four pens of 100 $\text{m}^2$ against a barn: $x = \sqrt{125} \approx 11.18$ m and $y = 100/\sqrt{125} \approx 8.94$ m.
- Common slips: stopping at the critical point; forgetting a dimension or the units.