Section 6.1: Absolute Extrema

The largest and smallest values of a function on an interval, and the Extreme Value Theorem

A motivating example: illicit drug use among college students

Example. Based on data from the National Institute on Drug Abuse, the percent y of college students who used an illicit drug between 1991 and 2011 can be approximated by

$$f(t) = .00331t^3 - .146t^2 + 1.96t + 28.2,$$

where t is the number of years since 1991.

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"Absolute max" means the *largest value on the whole interval* $0 \leq t \leq 20$, not just the top of a hill. Today: what that means, when it must exist, and how to find it.

Definition: absolute maximum and absolute minimum

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- $f(c)$ is the absolute maximum of $f(x)$ on the interval if $f(x) \leq f(c)$ for all x in the interval,
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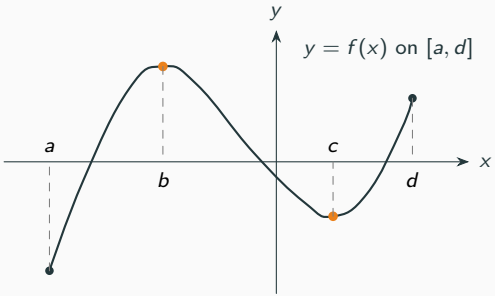
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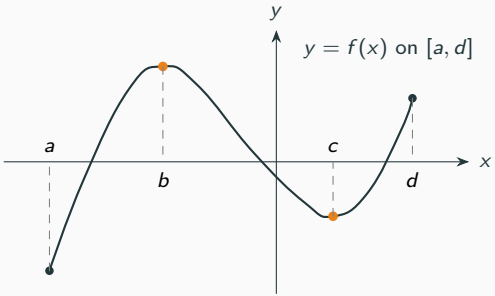
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- Absolute max = the **highest point** of the graph over the *whole* interval; absolute min = the **lowest point**.
- A *local* max only has to beat its neighbors; an *absolute* max has to beat *every* value on the interval.

Example: reading absolute extrema from a graph

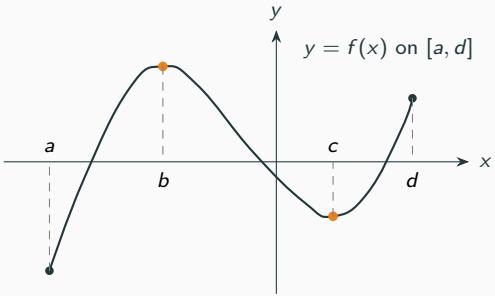


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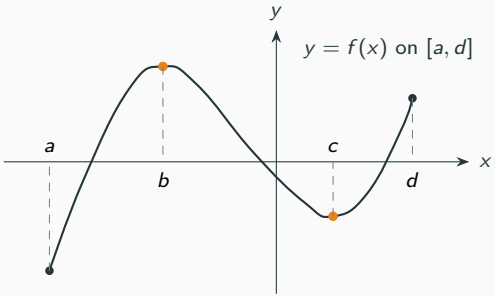
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$f(a)$ is the absolute min: the lowest point is the left **endpoint**, lower than the valley at c.

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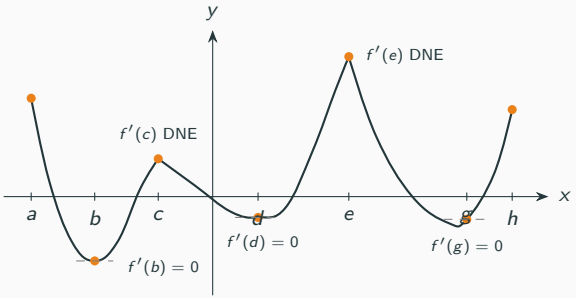


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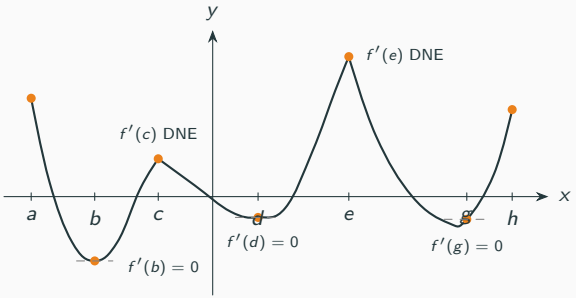
$f(a)$ is the absolute min: the lowest point is the left **endpoint**, lower than the valley at c.

So $f(c)$ is a local min but *not* the absolute min, and $f(d)$ is neither.

Example: absolute and local extrema together



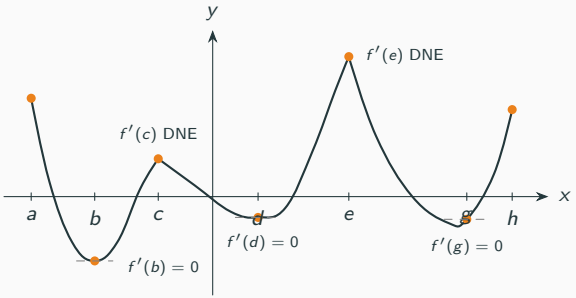
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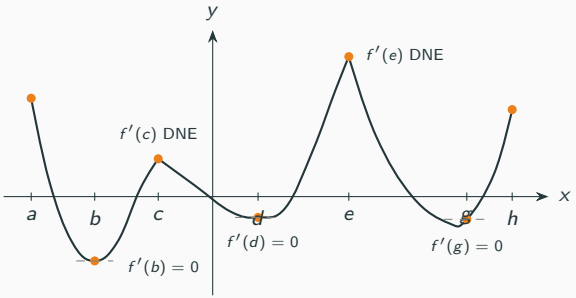
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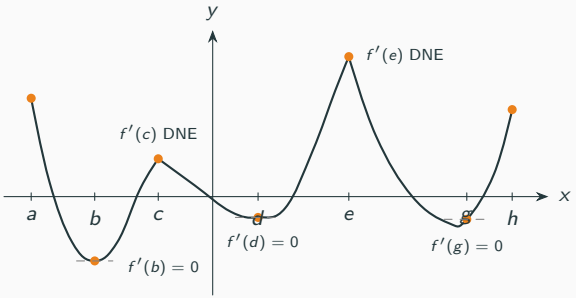


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The endpoints a and h give no extrema of any kind here.

Where can extrema occur?

Key difference

Absolute extrema can occur at **endpoints** or at **interior points** of the interval, but **local** extrema can only occur at **interior points** of the interval.

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- In the first graph the absolute min was at the endpoint a — an endpoint can never be a *local* extremum, since there is no graph on one side of it.
- At an interior point, an absolute extremum is automatically also a local extremum (like $f(e)$ and $f(b)$ in the second graph), so it sits at a **critical point**.

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Consequence

On an interval, the absolute extrema can only happen at critical points or at endpoints. That is the whole strategy for finding them.

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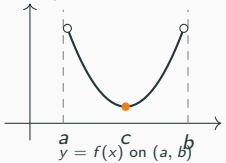
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The theorem promises the absolute max and min *exist*; it does not say where they are. The next slide shows what can go wrong when a hypothesis fails, and then we find them.

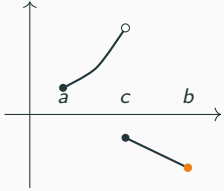
When the EVT does not apply

Open interval



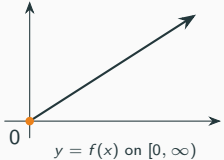
$y = f(x)$ on (a, b)

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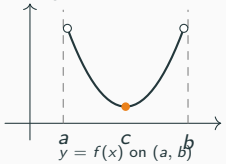
Interval NOT bounded



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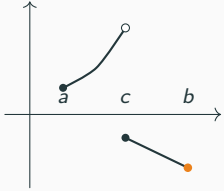
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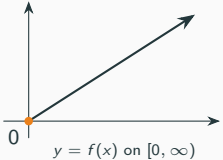
abs. min is $f(c)$,
but NO abs. max: the values climb toward the ends but never reach a largest one.

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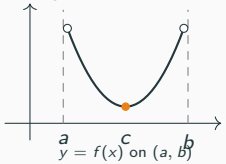
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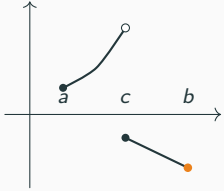
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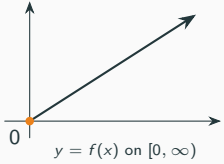
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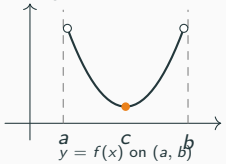
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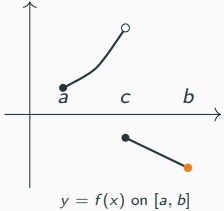
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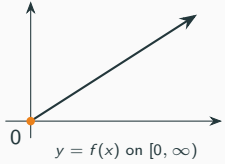
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Interval NOT bounded



abs. min is $f(0)$,
but NO abs. max: the values keep growing forever.

A theorem for intervals where the EVT does not apply

Theorem (one critical point)

Suppose $f(x)$ is continuous on an interval I and $f(x)$ has exactly one critical point $x = c$ in the interval I.

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- Why it works: with only one cp the graph can turn around only once — a single valley (or hill) on the whole interval.
- I may be open or unbounded, e.g. $(0, \infty)$. The theorem settles *one* absolute extremum; the other may not exist.

The procedure on a closed interval $[a, b]$

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No sign chart needed

Absolute extrema can only be at cps or endpoints, so the table of values decides everything — no need to classify the cps as local max or min.

Example (a): $f(x) = x^{8/3} - 16x^{2/3}$ on $[-1, 8]$

Example. Find the absolute extrema of $f(x)$ on the given interval. If there is none, explain why.

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Example (a), continued: critical points in $[-1, 8]$

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Where $f'(x) = 0$: $\quad 8x^2 - 32 = 0$

$8\left(x^2 - 4\right) = 0 \;\Rightarrow\; x = 2$ or $x = -2$

$x = 2$ is a cp (it is in $[-1, 8]$).

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Candidates

critical points $x = 0,\ 2$ and endpoints $x = -1,\ 8$.

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(Cube root first: $8^{1/3} = 2$, so $8^{8/3} = 2^8$, $8^{2/3} = 2^2$; and $(-1)^{1/3} = -1$, so both powers of -1 are 1.)

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cp	0	$f(0) = 0$
cp	2	$f(2) = 2^{8/3} - 16 \cdot 2^{2/3} \approx 6.35 - 25.40 = -19.05$ ← **smallest**

(Cube root first: $8^{1/3} = 2$, so $8^{8/3} = 2^8$, $8^{2/3} = 2^2$; and $(-1)^{1/3} = -1$, so both powers of -1 are 1.)

Example (a), continued: evaluate at the endpoints and cps

Evaluate $f(x)$ at the endpoints and cps to find the largest and smallest values:

	x	$f(x) = x^{8/3} - 16x^{2/3}$
endpoint	-1	$f(-1) = (-1)^{8/3} - 16(-1)^{2/3} = 1 - 16 = -15$
endpoint	8	$f(8) = 8^{8/3} - 16 \cdot 8^{2/3} = 2^8 - 16(4) = 256 - 64 = 192$ ← **biggest**
cp	0	$f(0) = 0$
cp	2	$f(2) = 2^{8/3} - 16 \cdot 2^{2/3} \approx 6.35 - 25.40 = -19.05$ ← **smallest**

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Answer

On $[-1, 8]$ the absolute max is $f(8) = 192$ and the absolute min is $f(2) \approx -19.05$.

Example (b): $f(x) = 2x + \frac{8}{x^2} + 1$ on $(0, \infty)$

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NOT closed and **NOT** bounded, so the EVT does NOT apply: there may or may not be absolute extrema. Use the one-critical-point theorem instead.

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Find $f'(x)$: $f'(x) = 2 - \dfrac{16}{x^3} = \dfrac{2x^3 - 16}{x^3} = \dfrac{2\left(x^3 - 8\right)}{x^3}$

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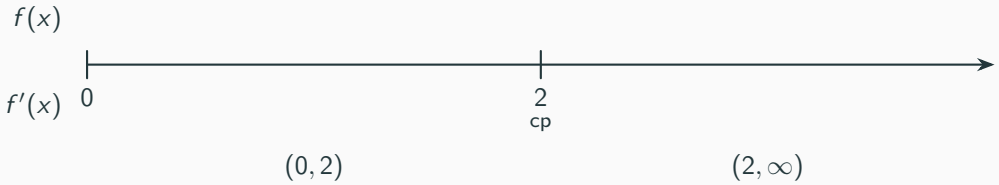
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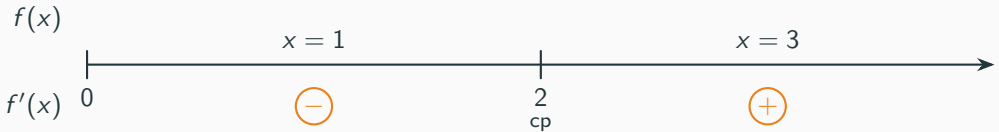
Cps of $f(x)$ in $(0, \infty)$:

- $f'(x) = 0$: $2\left(x^3 - 8\right) = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$ — the only cp.
- $f'(x)$ DNE: $x^3 = 0 \Rightarrow x = 0$, **NOT** in the interval.

Example (b), continued: sign chart and conclusion



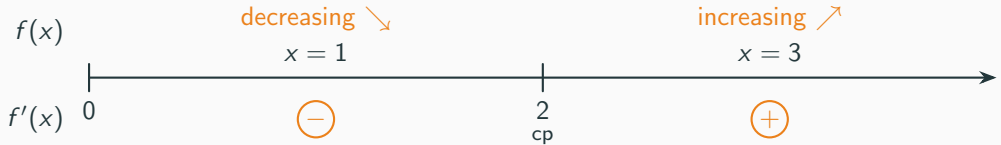
Example (b), continued: sign chart and conclusion



$(0, 2)$ $\qquad$ $(2, \infty)$

$f'(1) = \frac{2(1-8)}{1} = -14 < 0 \;\Rightarrow\; \ominus,$ $\qquad$ $f'(3) = \frac{2(27-8)}{27} = \frac{38}{27} > 0 \;\Rightarrow\; \oplus.$

Example (b), continued: sign chart and conclusion

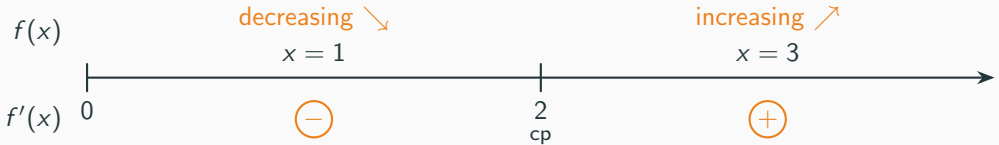


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Conclusion

f' goes $\ominus$ to $\oplus$ at $x = 2$: a local min. **By the theorem**, $f(2) = 4 + 2 + 1 = 7$ is the **absolute min** on $(0, \infty)$, since $x = 2$ is the only cp there.

Example (b), continued: sign chart and conclusion



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No absolute max: $f(x) \to \infty$ as $x \to 0^+$ ($\frac{8}{x^2}$ blows up), so no value is the largest.

Try it — Example (c): $f(x) = (6 - x)e^x$ on $[0, 6]$

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Find $f'(x)$ (product rule):

$$f'(x) = (-1)\, e^x + (6 - x)\, e^x$$

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Candidates: the cp $x = 5$ and the endpoints $x = 0$, $x = 6$.

Example (c), continued: evaluate at the endpoints and cps

Evaluate $f(x)$ at the endpoints and cps:

x	$f(x) = (6 - x)e^x$

Example (c), continued: evaluate at the endpoints and cps

Evaluate $f(x)$ at the endpoints and cps:

	x	$f(x) = (6-x)e^x$
cp	5	$f(5) = (6-5)e^5 = e^5 \approx 148.41$ ← **biggest**

Example (c), continued: evaluate at the endpoints and cps

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	x	$f(x) = (6-x)e^x$
cp	5	$f(5) = (6-5)e^5 = e^5 \approx 148.41$ ← **biggest**
endpoint	0	$f(0) = (6-0)e^0 = 6$

Example (c), continued: evaluate at the endpoints and cps

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cp	5	$f(5) = (6-5)e^5 = e^5 \approx 148.41$ ← **biggest**
endpoint	0	$f(0) = (6-0)e^0 = 6$
endpoint	6	$f(6) = (6-6)e^6 = 0$ ← **smallest**

Example (c), continued: evaluate at the endpoints and cps

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Answer

$f(5) = e^5$ is the absolute max of $f(x)$ on $[0, 6]$, and $f(6) = 0$ is the absolute min of $f(x)$ on $[0, 6]$.

Example (c), continued: evaluate at the endpoints and cps

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Answer

$f(5) = e^5$ is the absolute max of $f(x)$ on $[0, 6]$, and $f(6) = 0$ is the absolute min of $f(x)$ on $[0, 6]$.

(The exact value e^5 is the answer; ≈ 148.41 just confirms it is the biggest.)

Back to the drug-use model: set-up

$f(t) = .00331t^3 - .146t^2 + 1.96t + 28.2$, $t =$ years since 1991; 1991 to 2011 means $0 \leq t \leq 20$.

EVT applies

f is a polynomial (continuous) on the closed, bounded interval $[0, 20]$, so the absolute max and min exist.

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Cps of $f(t)$: $f'(t)$ is a polynomial (never undefined), so set it equal to 0:

$.00993t^2 - .292t + 1.96 = 0$ — a quadratic: use the quadratic formula.

Drug-use model, continued: solve for the critical points

Quadratic formula with $a = .00331(3) = .00993$, $b = -.146(2) = -.292$, $c = 1.96$:

$$t = \frac{.146(2) \pm \sqrt{(-.146(2))^2 - 4\,(.00331(3))\,(1.96)}}{2\,(.00331(3))}$$

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Two critical points, both in $[0, 20]$

$t \approx \dfrac{.3781}{.01986} \approx 19.04$ or $t \approx \dfrac{.2059}{.01986} \approx 10.37$

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Candidates: the cps $t \approx 10.37,\ 19.04$ and the endpoints $t = 0,\ 20$.

Drug-use model, continued: evaluate $f(t)$ at the cps and endpoints

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cp	10.37	$f(10.37) \approx 36.52 \leftarrow$ **biggest**

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Answer

About 10.37 years after 1991 (during 2001), illicit drug use by college students reached its absolute max of about **36.5**% in the given period.

Drug-use model, continued: evaluate $f(t)$ at the cps and endpoints

	t	$f(t) = .00331t^3 - .146t^2 + 1.96t + 28.2$
endpoint	0	$f(0) = 28.2 \leftarrow$ **smallest**
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Answer

About 10.37 years after 1991 (during 2001), illicit drug use by college students reached its absolute max of about **36.5**% in the given period.

Note: the minimum use of 28.2% occurred in 1991 ($t = 0$, an endpoint).

Summary

Absolute extrema

$f(c)$ is the absolute max (min) on an interval if $f(x) \le f(c)$ ($f(x) \ge f(c)$) for *all* x in the interval. They can occur at endpoints or interior points; local extrema only at interior points (cps).

Extreme Value Theorem and the procedure

Continuous f on a closed, bounded $[a, b] \Rightarrow$ absolute max and min exist. Evaluate f at the cps inside $[a, b]$ and at the endpoints: largest = abs. max, smallest = abs. min.
If the EVT does not apply there may be none; but if f is continuous with *exactly one* cp, a local max/min there is absolute.

- $x^{8/3} - 16x^{2/3}$ on $[-1, 8]$: max $f(8) = 192$, min $f(2) \approx -19.05$. $(6 - x)e^x$ on $[0, 6]$: max e^5, min 0.
- $2x + \frac{8}{x^2} + 1$ on $(0, \infty)$: abs. min $f(2) = 7$ (one cp), no abs. max.
- Drug-use model on $[0, 20]$: absolute max $\approx 36.5\%$ at $t \approx 10.37$ (2001); minimum 28.2% in 1991.