Section 5.3: Concavity and the Second Derivative

Higher derivatives, inflection points, and the second derivative test

Notation: the second derivative

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The second derivative of $y = f(x)$ is the derivative of $f'(x)$. It can be written as

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Today

Read the bend (concavity) from f'', find where it switches (inflection points), and use f'' to classify local extrema.

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concave up
(can hold water)



concave down
(can't hold water)

Concave up: the tangent slopes go from negative to positive — the slope is increasing, and the graph bends upward.

Concave down: the slopes decrease, and the graph bends downward.

The second derivative and concavity

Since concavity is about whether f' is increasing or decreasing, we test it with the derivative of f':

Theorem

Suppose $f(x)$ is twice differentiable on an open interval I. Then

1. If $f''(x) > 0$ for all x in I, then $f(x)$ is concave up on I.

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Compare with last time

$f' > 0$ means f is increasing; $f'' > 0$ means f' is increasing, i.e. f is concave up.
The same sign-chart method works — we just test $f''(x)$ instead of $f'(x)$.

Definition: inflection points

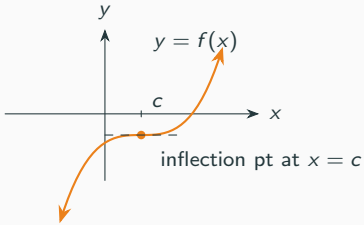
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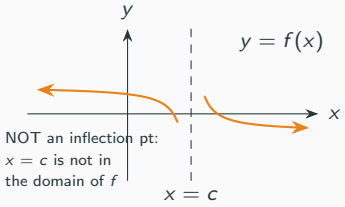
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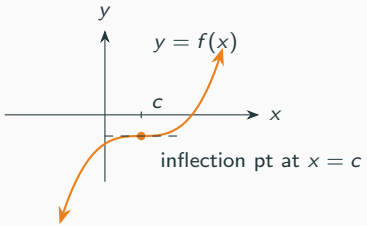


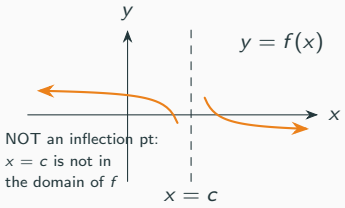


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Right: the concavity changes across $x = c$, but there is no point on the graph there — not an inflection *point*.

Strategy: concavity and inflection points

Where to look

At an inflection point $x = c$ of $f(x)$, $f''(c) = 0$ or $f''(c)$ DNE. So the **candidates** are the "cps of f''" — just as the cps of f were the candidates for local extrema.

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3. **Sign chart for** f'': one test point per interval. $\oplus$: concave up $\smile$; $\ominus$: concave down $\frown$.
4. **Inflection points**: where the sign of f'' *changes*, at a point *in the domain* of f.

The second derivative and local extrema

The second derivative test

Suppose $f(x)$ is twice differentiable on an open interval containing c.

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local max at $x = c$.

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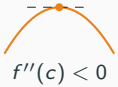
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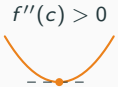
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Concave down at a horizontal tangent means the top of a hill; concave up means the bottom of a valley.

Note: if $f''(c) = 0$ the test says nothing — go back to the first derivative test.

Example: $f(x) = \frac{x-8}{x^2}$ — Step 1: find $f''(x)$

Example. Given $f(x) = \dfrac{x-8}{x^2}$, find the intervals where $f(x)$ is concave up and concave down, and find the x-coordinate of all inflection points of $f(x)$.

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Step 1: find $f''(x)$ — from last time, $f'(x) = \dfrac{-x+16}{x^3}$

$$f''(x) = \frac{(-1)\cdot x^3 - (-x+16)\cdot 3x^2}{(x^3)^2}$$

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$$f''(x) = \frac{(-1)\cdot x^3 - (-x+16)\cdot 3x^2}{(x^3)^2}$$
$$= \frac{-x^3 + 3x^3 - 48x^2}{x^6} = \frac{2x^3 - 48x^2}{x^6}$$

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$$\begin{aligned} f''(x) &= \frac{(-1)\cdot x^3 - (-x+16)\cdot 3x^2}{(x^3)^2} \\ &= \frac{-x^3 + 3x^3 - 48x^2}{x^6} = \frac{2x^3 - 48x^2}{x^6} \\ &= \frac{2x^2(x-24)}{x^6} = \frac{2(x-24)}{x^4} \qquad \text{(cancel } x^2\text{)} \end{aligned}$$

Example, Step 2: candidates for inflection points

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Where $f''(x) = 0$ (numerator $= 0$):

$$x - 24 = 0 \quad \Longrightarrow \quad x = 24$$

24 is in the domain of f, so it is the **only** candidate.

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Where $f''(x)$ **DNE** (denominator $= 0$):

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0 is **NOT** in the domain of f or f' (it is a VA), so it cannot be an inflection point.

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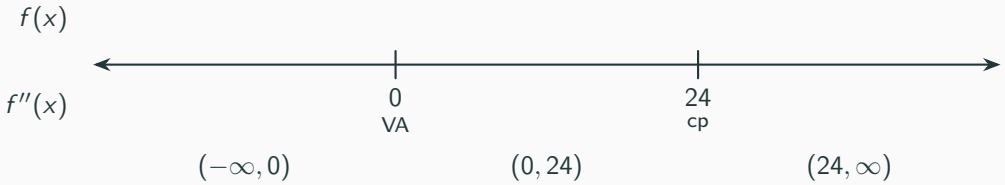
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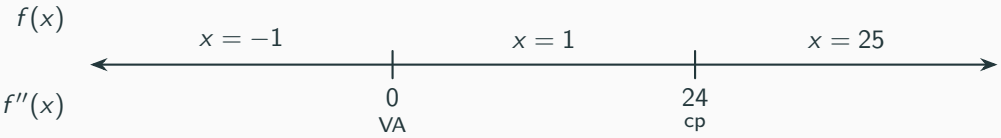
As before

$x = 0$ still splits the number line: the sign of f'' can change across a VA.

Example, Step 3: sign chart for $f''(x)$

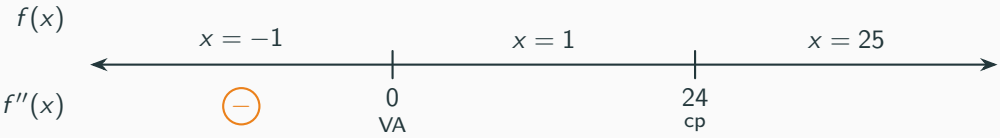


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Test one point per interval in $f''(x) = \frac{2(x-24)}{x^4}$ (its denominator x^4 is always positive):

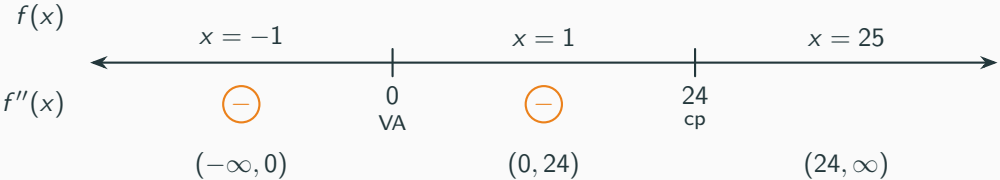
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Test one point per interval in $f''(x) = \frac{2(x-24)}{x^4}$ (its denominator x^4 is always positive):

$$x = -1: \quad f''(-1) = \frac{2(-1-24)}{(-1)^4} = \frac{-50}{1} < 0 \qquad \Rightarrow \ \ominus$$

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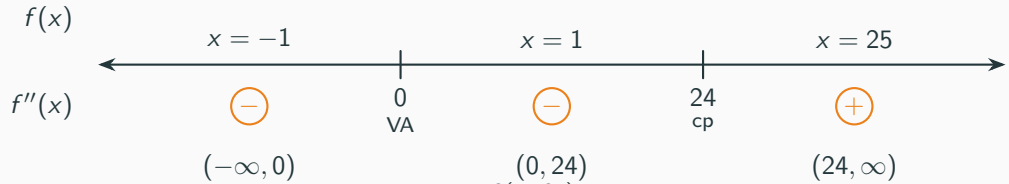


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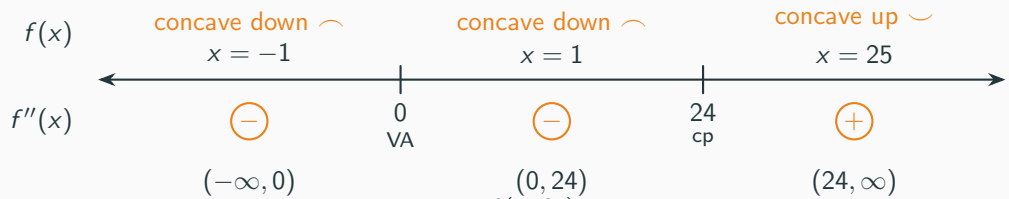
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Example, Step 4: read off the answer

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$f(x)$ is concave down on $(-\infty, 0)$ and $(0, 24)$, and concave up on $(24, \infty)$.

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- $x = 0$ is not an inflection point: the sign of f'' does not even change there, and in any case $f(0)$ is undefined.
- Only the x-coordinate was asked for; the inflection point itself is $(24, f(24)) = \left(24, \frac{16}{576}\right) = \left(24, \frac{1}{36}\right)$.

Example, continued: the second derivative test at $x = 16$

Strategy: classifying a cp with the second derivative test

1. Find the cps where $f'(c) = 0$.

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Strategy: classifying a cp with the second derivative test

1. Find the cps where $f'(c) = 0$.
2. Evaluate $f''(c)$ — only its sign matters.
3. $f''(c) < 0$: local max. $f''(c) > 0$: local min. $f''(c) = 0$, or $f'(c)$ DNE: no conclusion — use the first derivative test instead.

Example, continued: the second derivative test at $x = 16$

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For $f(x) = \frac{x-8}{x^2}$: last time the only cp was $x = 16$, where $f'(16) = 0$. With $f''(x) = \frac{2(x-24)}{x^4}$:

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That is the answer the first derivative test gave last time, with no sign chart for f' needed. (Only the sign chart for f' gives the intervals of increase and decrease, though.)

Try it: $f(x) = \frac{x}{x^2-4}$

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Given $f(x) = \frac{x}{x^2-4}$, $f'(x) = \frac{-(x^2+4)}{(x^2-4)^2}$ and $f''(x) = \frac{2x\left(x^2+12\right)}{(x^2-4)^3}$, find the intervals where $f(x)$ is concave up and down, and the x-coordinate of all inflection points of $f(x)$.

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Step 1 is done (f'' is given). **Step 2: the candidates (cps of f').**

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Where $f''(x) = 0$: $2x\left(x^2+12\right) = 0$

$2x = 0 \Rightarrow x = 0$

or $x^2 + 12 = 0 \Rightarrow x^2 = -12$ NOT real

$x = 0$ is in the domain: the **only** candidate.

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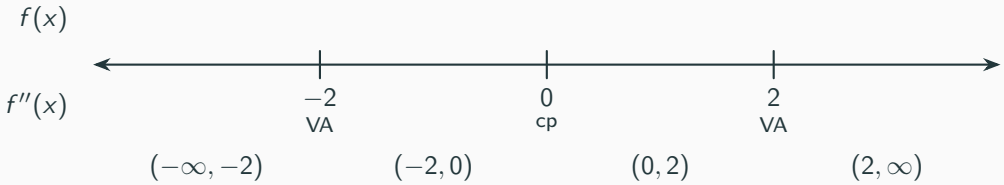
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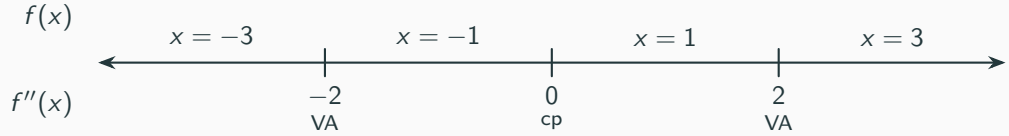
$x^2 - 4 = 0 \Rightarrow x = \pm 2$

NOT in the domain of f or f' (VAs): not inflection points, but they split the number line.

Try it, Step 3: sign chart for $f''(x)$

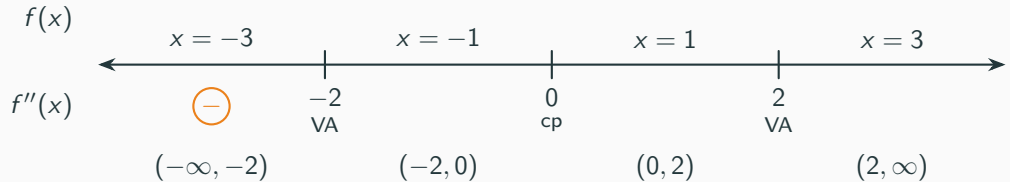


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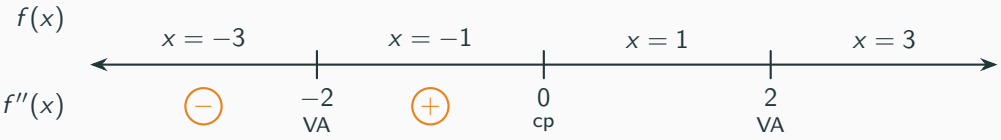
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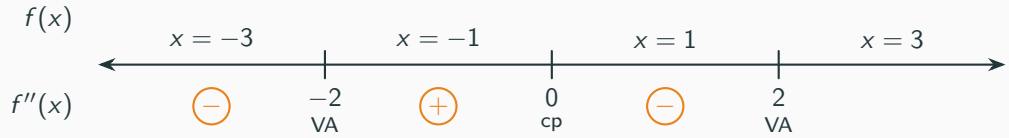
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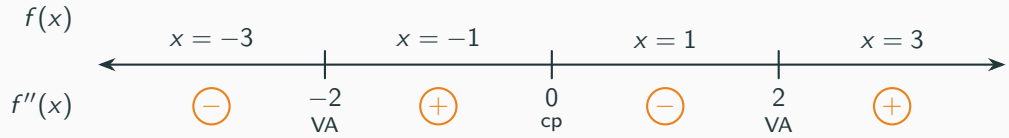


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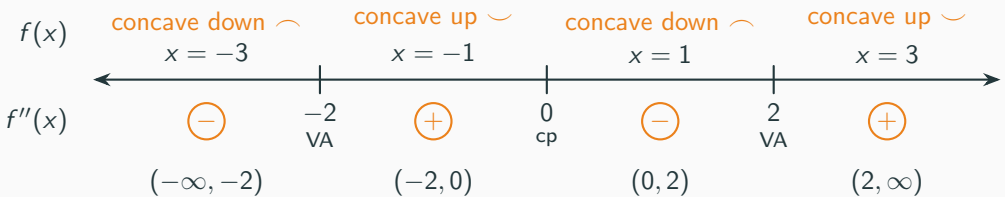
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Try it, Step 4: the answer

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$f(x)$ is concave down on $(-\infty, -2)$ and $(0, 2)$, and concave up on $(-2, 0)$ and $(2, \infty)$.

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(The point is $(0, f(0)) = (0, 0)$.)

What about $x = \pm 2$?

The concavity changes across ± 2, but f is not defined there (VAs), so there is no inflection *point* at ± 2.

Try it: the point of inflection of the population curve

Try it

Suppose the size of a population at time t is $N(t) = \dfrac{100}{1 + 3e^{-2t}}$ for $t \geq 0$. Find the point of inflection $\big(t, N(t)\big)$ of $N(t)$.

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From last time: $N'(t) = \frac{600e^{-2t}}{(1+3e^{-2t})^2}$. **Step 1: find** $N''(t)$ with the quotient rule. The pieces:

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- $\frac{d}{dt}\left[600e^{-2t}\right] = 600 \cdot (-2)e^{-2t} = -1200e^{-2t}$
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$$N''(t) = \frac{-1200e^{-2t}\left(1+3e^{-2t}\right)^2 - 600e^{-2t} \cdot 2\left(1+3e^{-2t}\right)\left(-6e^{-2t}\right)}{\left(1+3e^{-2t}\right)^4}$$

Population, Step 1 (continued): simplify $N''(t)$

Both terms of the numerator contain $1200e^{-2t}\left(1+3e^{-2t}\right)$ (since $600 \cdot 2 = 1200$), so factor it out:

$$N''(t) = \frac{1200e^{-2t}\left(1+3e^{-2t}\right)\left[-\left(1+3e^{-2t}\right)+6e^{-2t}\right]}{\left(1+3e^{-2t}\right)^4}$$

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&= \frac{1200e^{-2t}\left(-1+3e^{-2t}\right)}{\left(1+3e^{-2t}\right)^3} \qquad \text{this is } N''(t)
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Population, Step 2: candidates for the inflection point

Where $N''(t) = 0$: $1200e^{-2t}\left(-1 + 3e^{-2t}\right) = 0$. The factor $1200e^{-2t}$ is NEVER 0 (an exponential is never 0), so we need $-1 + 3e^{-2t} = 0$:

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Where $N''(t)$ DNE: $\left(1 + 3e^{-2t}\right)^3 = 0 \;\Rightarrow\; 3e^{-2t} = -1 \;\Rightarrow\; e^{-2t} = -\frac{1}{3}$ NEVER (an exponential is never negative).

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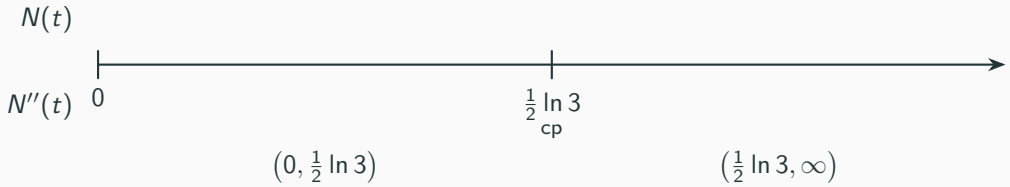
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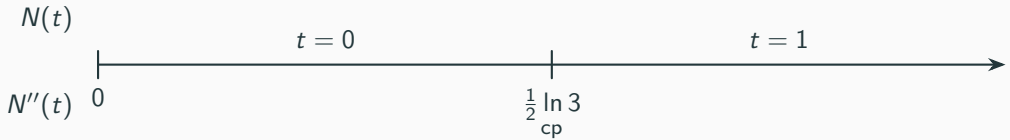
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So $t = \frac{1}{2}\ln(3)$ is the only cp of $N'(t)$ — the only candidate for an inflection point.

Population, Step 3: sign chart for $N''(t)$

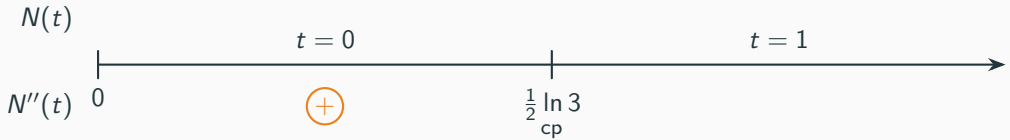


Population, Step 3: sign chart for $N''(t)$



In $N''(t)$ the factor $1200e^{-2t}$ and the denominator $\left(1 + 3e^{-2t}\right)^3$ are always positive, so the sign of $N''(t)$ is the sign of the middle factor $-1 + 3e^{-2t}$:

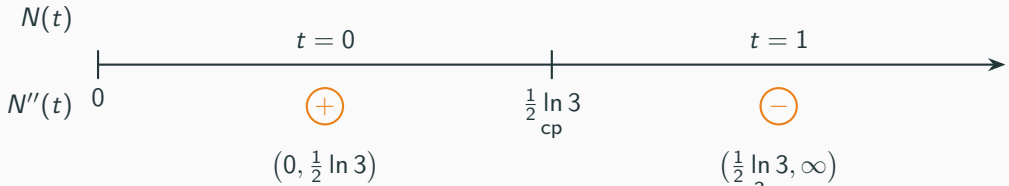
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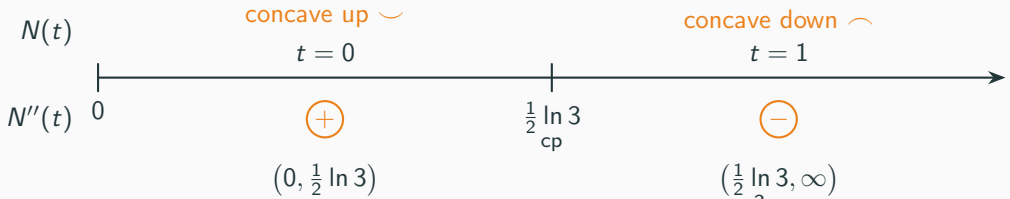


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Population, Step 4: the point of inflection

t-coordinate: $\frac{1}{2}\ln 3$. N-coordinate (use $e^{-\ln 3} = e^{\ln(1/3)} = \frac{1}{3}$):

$$N\left(\tfrac{1}{2}\ln 3\right) = \frac{100}{1+3e^{-2\cdot\frac{1}{2}\ln 3}} = \frac{100}{1+3e^{-\ln 3}}$$

Population, Step 4: the point of inflection

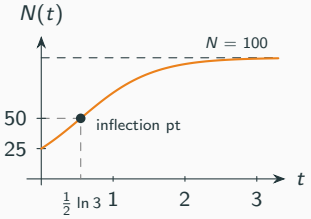
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$$= \frac{100}{1+3\cdot\frac{1}{3}} = \frac{100}{2} = 50$$

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$$= \frac{100}{1+3\cdot\frac{1}{3}} = \frac{100}{2} = 50$$



Answer

$\left(\frac{1}{2}\ln 3,\ 50\right) \approx (0.55,\ 50)$ is the point of inflection: **half the carrying capacity** 100, and where the population grows *fastest*. We'll revisit this again in Section 11.2!

Summary

The strategy for concavity and inflection points

1. Find $f''(x)$ (the derivative of $f'(x)$).
2. Candidates: where $f''(x) = 0$ or $f''(x)$ DNE (the "cps of f'"); zeros of the denominator outside the domain are VAs, not inflection points — but they still split the number line.
3. Sign chart for f'': $\oplus$ concave up (holds water), $\ominus$ concave down.
4. An inflection point is a point *in the domain* where the sign of f'' changes.

Second derivative test

If $f'(c) = 0$: $f''(c) < 0$ gives a local max, $f''(c) > 0$ gives a local min, $f''(c) = 0$ is inconclusive.

- $\frac{x-8}{x^2}$: inflection point at $x = 24$. $\frac{x}{x^2-4}$: inflection point at $x = 0$ (not at the VAs ± 2).
- Logistic curve $N(t) = \frac{100}{1+3e^{-2t}}$: inflection point $\left(\frac{1}{2}\ln 3, 50\right)$, at half the carrying capacity.