Section 4.3: The Chain Rule

Derivative of the outside, evaluated at the inside, times the derivative of the inside

A motivating example: the Tasmanian devil

The Tasmanian devil often preys on animals larger than itself. Researchers have identified the mathematical relationship

$$L(w) = 2.265\, w^{2.543},$$

where w (in kg) is the weight and $L(w)$ (in mm) is the length. Suppose the weight of a particular Tasmanian devil can be estimated by

$$w(t) = 0.125 + 0.18\, t,$$

where $w(t)$ is the weight (in kg) and t is the age (in weeks) of a Tasmanian devil that is less than one year old.

Try it

How fast is the **length** of a 30-week-old Tasmanian devil changing?

The Tasmanian devil: chaining the two rates together

$$\frac{dL}{dw} = (2.265)(2.543)\, w^{2.543-1} \approx 5.76\, w^{1.543}$$

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$$\frac{dL}{dt} = 5.76\,\big(0.125 + 0.18(30)\big)^{1.543}\,(0.18) \approx 14.5$$

Interpretation

So the length of a 30-week-old Tasmanian devil is increasing at a rate of about **14.5 mm/week**.

The Tasmanian devil: another route to the same answer

Note

Can also substitute $w(t)$ into $L(w)$ *first*,

$$L(t) = 2.265\left(0.125 + 0.18\,t\right)^{2.543},$$

and *then* take the derivative with respect to t of $L(t)$.

The Tasmanian devil: another route to the same answer

Note

Can also substitute $w(t)$ into $L(w)$ *first*,

$$L(t) = 2.265\left(0.125 + 0.18\,t\right)^{2.543},$$

and *then* take the derivative with respect to t of $L(t)$.

But $L(t)$ is a function *of* a function — $2.265(\,\cdot\,)^{2.543}$ applied to $0.125 + 0.18t$ — and we do not yet have a rule for differentiating that. That rule is the chain rule.

The Chain Rule

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Let $h = f(g(x))$, where $f(x)$ and $g(x)$ are differentiable. Then

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

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In words

"derivative of **outside** evaluated at **inside** times derivative of **inside**"

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Let $h = f(g(x))$, where $f(x)$ and $g(x)$ are differentiable. Then

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

In words

"derivative of **outside** evaluated at **inside** times derivative of **inside**"

How to spot it

A chain-rule problem is a function *of* something other than plain x: "sin of $2x^2$," "ln of $x^3 + x$," "e to the $4x$."

Calculate $f'(x)$: (a) $f(x) = e^{4x}$

Try it: calculate $f'(x)$ for $f(x) = e^{4x}$.

Calculate $f'(x)$: **(a)** $f(x) = e^{4x}$

Try it: calculate $f'(x)$ for $f(x) = e^{4x}$. Chain

$$\text{outside} = e^{\color{orange}x} \xrightarrow{\frac{d}{dx}} e^{\color{orange}x}$$

$$\text{inside} = \color{orange}4x \xrightarrow{\frac{d}{dx}} \underbrace{\frac{d}{dx}[4x]}_{c\,x^n} = 4\,\frac{d}{dx}\left[x^1\right] = 4 \cdot 1x^{1-1} = 4$$

Calculate $f'(x)$: **(a)** $f(x) = e^{4x}$

Try it: calculate $f'(x)$ for $f(x) = e^{4x}$.

Chain

$$\text{outside} = e^{x} \xrightarrow{\frac{d}{dx}} e^{x}$$

$$\text{inside} = 4x \xrightarrow{\frac{d}{dx}} \underbrace{\frac{d}{dx}[4x]}_{c\,x^n} = 4\,\frac{d}{dx}\left[x^1\right] = 4 \cdot 1x^{1-1} = 4$$

$$f'(x) = \underbrace{e^{4x}}_{\text{outside' at inside}} \cdot \underbrace{4}_{\text{inside'}} = e^{4x} \cdot 4$$

Calculate $f'(x)$: **(b)** $f(x) = \sqrt{x+2}$

Try it: calculate $f'(x)$ for $f(x) = \sqrt{x+2}$.

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Try it: calculate $f'(x)$ for $f(x) = \sqrt{x+2}$.

Chain

$$\text{outside} = \sqrt{x} \qquad \xrightarrow{\frac{d}{dx}} \frac{d}{dx}\left[x^{1/2}\right] = \tfrac{1}{2}\,x^{-1/2}$$

$$\text{inside} = x+2 \qquad \xrightarrow{\frac{d}{dx}} \frac{d}{dx}[x] + \frac{d}{dx}[2] = 1 + 0$$

Calculate $f'(x)$: **(b)** $f(x) = \sqrt{x+2}$

Try it: calculate $f'(x)$ for $f(x) = \sqrt{x+2}$.

Chain

$$\text{outside} = \sqrt{x} \qquad \xrightarrow{\frac{d}{dx}} \frac{d}{dx}\left[x^{1/2}\right] = \tfrac{1}{2}\,x^{-1/2}$$

$$\text{inside} = x+2 \qquad \xrightarrow{\frac{d}{dx}} \frac{d}{dx}[x] + \frac{d}{dx}[2] = 1 + 0$$

$$f'(x) = \underbrace{\tfrac{1}{2}\,(x+2)^{-1/2}}_{\text{outside}' \text{ at inside}} \cdot \underbrace{(1)}_{\text{inside}'} = \tfrac{1}{2}\,(x+2)^{-1/2}\,(1)$$

Calculate $f'(x)$: **(c)** $f(x) = \sin(2x^2)$

Try it: calculate $f'(x)$ for $f(x) = \sin(2x^2)$.

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Try it: calculate $f'(x)$ for $f(x) = \sin(2x^2)$.

"sin of $2x^2$" $\Rightarrow$ **Chain**

$$\text{outside} = \sin(x) \xrightarrow{\frac{d}{dx}} \cos(x)$$

$$\text{inside} = 2x^2 \xrightarrow{\frac{d}{dx}} \underbrace{\frac{d}{dx}\left[2x^2\right]}_{c\,x^n} = 2\,\frac{d}{dx}\left[x^2\right] = 4x$$

Calculate $f'(x)$: **(c)** $f(x) = \sin(2x^2)$

Try it: calculate $f'(x)$ for $f(x) = \sin(2x^2)$. "sin of $2x^2$" $\Rightarrow$ **Chain**

$$\text{outside} = \sin(x) \xrightarrow{\frac{d}{dx}} \cos(x)$$

$$\text{inside} = 2x^2 \xrightarrow{\frac{d}{dx}} \underbrace{\frac{d}{dx}\left[2x^2\right]}_{c\,x^n} = 2\,\frac{d}{dx}\left[x^2\right] = 4x$$

$$f'(x) = \underbrace{\cos(2x^2)}_{\text{outside' at inside}} \cdot \underbrace{4x}_{\text{inside'}} = \cos(2x^2) \cdot 4x$$

Calculate $f'(x)$: **(d)** $f(x) = \ln(x^3 + x)$

Try it: calculate $f'(x)$ for $f(x) = \ln(x^3 + x)$.

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"ln of $x^3 + x$" $\Rightarrow$ **Chain**

$$\text{outside} = \ln(x) \xrightarrow{\frac{d}{dx}} \frac{1}{x}$$

$$\text{inside} = x^3 + x \xrightarrow{\frac{d}{dx}} 3x^2 + 1$$

Calculate $f'(x)$: **(d)** $f(x) = \ln(x^3 + x)$

Try it: calculate $f'(x)$ for $f(x) = \ln(x^3 + x)$.

"ln of $x^3 + x$" $\Rightarrow$ **Chain**

$$\text{outside} = \ln(x) \xrightarrow{\frac{d}{dx}} \frac{1}{x}$$

$$\text{inside} = x^3 + x \xrightarrow{\frac{d}{dx}} 3x^2 + 1$$

$$f'(x) = \underbrace{\frac{1}{x^3 + x}}_{\text{outside' at inside}} \cdot \underbrace{(3x^2 + 1)}_{\text{inside'}} = \frac{3x^2 + 1}{x^3 + x}$$

Calculate $f'(x)$: **(e)** $f(x) = 2x^4 \cos(3e^x)$

Try it: calculate $f'(x)$ for $f(x) = 2x^4 \cos(3e^x)$.

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Product

$$f'(x) = \frac{d}{dx}\left[2x^4\right] \cos(3e^x) + 2x^4 \frac{d}{dx}[\cos(3e^x)]$$

Calculate $f'(x)$: (e) $f(x) = 2x^4 \cos(3e^x)$

Try it: calculate $f'(x)$ for $f(x) = 2x^4 \cos(3e^x)$. **Product**

$$f'(x) = \frac{d}{dx}\left[2x^4\right] \cos(3e^x) + 2x^4 \, \frac{d}{dx}[\cos(3e^x)]$$

The pieces

$$\underbrace{\frac{d}{dx}\left[2x^4\right]}_{c\,x^n} = 2 \underbrace{\frac{d}{dx}\left[x^4\right]}_{x^n} = 8x^3 \qquad \frac{d}{dx}[\cos(3e^x)] = -\sin(3e^x) \cdot 3e^x$$ **Chain**

chain piece: outside $= \cos(x) \xrightarrow{\frac{d}{dx}} -\sin(x)$, inside $= 3e^x \xrightarrow{\frac{d}{dx}} \frac{d}{dx}[3e^x] = 3\,\frac{d}{dx}[e^x] = 3e^x$

Calculate $f'(x)$: (e) $f(x) = 2x^4 \cos(3e^x)$

Try it: calculate $f'(x)$ for $f(x) = 2x^4 \cos(3e^x)$. **Product**

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[2x^4\right]\cos(3e^x) + 2x^4\,\frac{d}{dx}[\cos(3e^x)] \\ &= 8x^3\cos(3e^x) + 2x^4\big(-\sin(3e^x)\cdot 3e^x\big) \end{aligned}$$

The pieces

$$\underbrace{\frac{d}{dx}\left[2x^4\right]}_{c\,x^n} = 2\,\underbrace{\frac{d}{dx}\left[x^4\right]}_{x^n} = 8x^3 \qquad \frac{d}{dx}[\cos(3e^x)] = -\sin(3e^x)\cdot 3e^x \quad \textbf{Chain}$$

chain piece: outside $= \cos(x) \xrightarrow{\frac{d}{dx}} -\sin(x)$, inside $= 3e^x \xrightarrow{\frac{d}{dx}} \frac{d}{dx}[3e^x] = 3\,\frac{d}{dx}[e^x] = 3e^x$

Calculate $f'(x)$: **(f)** $f(x) = \frac{\ln(x^3+30)}{8x}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{\ln(x^3 + 30)}{8x}$.

Calculate $f'(x)$: **(f)** $f(x) = \frac{\ln(x^3+30)}{8x}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{\ln(x^3+30)}{8x}$.

Quotient

$$f'(x) = \frac{\frac{d}{dx}\left[\ln(x^3+30)\right]\ 8x - \ln(x^3+30)\ \frac{d}{dx}[8x]}{(8x)^2}$$

Calculate $f'(x)$: **(f)** $f(x) = \frac{\ln(x^3+30)}{8x}$

Try it: calculate $f'(x)$ for $f(x) = \dfrac{\ln(x^3+30)}{8x}$. **Quotient**

$$f'(x) = \frac{\frac{d}{dx}\left[\ln(x^3+30)\right]\ 8x - \ln(x^3+30)\ \frac{d}{dx}[8x]}{(8x)^2}$$

The pieces

$$\frac{d}{dx}\left[\ln(x^3+30)\right] = \frac{1}{x^3+30}\cdot 3x^2 = \frac{3x^2}{x^3+30}$$ **Chain**

$$\underbrace{\frac{d}{dx}[8x]}_{c\,x^n} = 8\frac{d}{dx}[x] = 8(1) = 8$$

chain piece: outside $= \ln(x) \xrightarrow{\frac{d}{dx}} \frac{1}{x}$, inside $= x^3+30 \xrightarrow{\frac{d}{dx}} \frac{d}{dx}\left[x^3\right] + \frac{d}{dx}[30] = 3x^2+0$

Calculate $f'(x)$: **(f)** $f(x) = \frac{\ln(x^3+30)}{8x}$, **continued**

$$f'(x) = \frac{\frac{d}{dx}\left[\ln(x^3+30)\right]\ 8x - \ln(x^3+30)\ \frac{d}{dx}[8x]}{(8x)^2}$$

Calculate $f'(x)$: **(f)** $f(x) = \frac{\ln(x^3+30)}{8x}$, **continued**

$$f'(x) = \frac{\frac{d}{dx}\left[\ln(x^3+30)\right]\ 8x - \ln(x^3+30)\,\frac{d}{dx}[8x]}{(8x)^2}$$

$$= \frac{\left(\frac{3x^2}{x^3+30}\right)(8x) - \left(\ln(x^3+30)\right)(8)}{(8x)^2}$$

Calculate $f'(x)$: **(g)** $f(x) = \tan\left(\frac{\ln(x)}{2x-5}\right)$

Try it: calculate $f'(x)$ for $f(x) = \tan\left(\dfrac{\ln(x)}{2x-5}\right)$.

Calculate $f'(x)$: **(g)** $f(x) = \tan\left(\frac{\ln(x)}{2x-5}\right)$

Try it: calculate $f'(x)$ for $f(x) = \tan\left(\frac{\ln(x)}{2x-5}\right)$. **Chain**

$$\text{outside} = \tan(x) \xrightarrow{\frac{d}{dx}} \sec^2(x)$$

$$\text{inside} = \frac{\ln(x)}{2x-5} \text{ (Quotient)} \xrightarrow{\frac{d}{dx}} \frac{\frac{d}{dx}[\ln(x)]\,(2x-5) - \ln(x)\,\frac{d}{dx}[2x-5]}{(2x-5)^2}$$

Calculate $f'(x)$: **(g)** $f(x) = \tan\left(\frac{\ln(x)}{2x-5}\right)$

Try it: calculate $f'(x)$ for $f(x) = \tan\left(\frac{\ln(x)}{2x-5}\right)$. **Chain**

$$\text{outside} = \tan(x) \xrightarrow{\frac{d}{dx}} \sec^2(x)$$

$$\text{inside} = \frac{\ln(x)}{2x-5} \text{ (Quotient)} \xrightarrow{\frac{d}{dx}} \frac{\frac{d}{dx}[\ln(x)](2x-5) - \ln(x)\frac{d}{dx}[2x-5]}{(2x-5)^2}$$

$$= \frac{\frac{1}{x}\cdot(2x-5) - \ln(x)\cdot 2}{(2x-5)^2}$$

Calculate $f'(x)$: **(g)** $f(x) = \tan\left(\frac{\ln(x)}{2x-5}\right)$, **continued**

Derivative of the outside, evaluated at the inside, times the derivative of the inside:

$$f'(x) = \sec^2\left(\frac{\ln(x)}{2x-5}\right)\left[\frac{\frac{1}{x}\cdot(2x-5)-\ln(x)\cdot 2}{(2x-5)^2}\right]$$

Calculate $f'(x)$: **(h)** $f(x) = x^{12}e^{5-3x}$

Try it: calculate $f'(x)$ for $f(x) = x^{12}e^{5-3x}$.

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Try it: calculate $f'(x)$ for $f(x) = x^{12}e^{5-3x}$.

Product

$$f'(x) = \frac{d}{dx}\left[x^{12}\right]\ e^{5-3x} + x^{12}\ \frac{d}{dx}\left[e^{5-3x}\right]$$

Calculate $f'(x)$: **(h)** $f(x) = x^{12}e^{5-3x}$

Try it: calculate $f'(x)$ for $f(x) = x^{12}e^{5-3x}$. **Product**

$$f'(x) = \frac{d}{dx}\left[x^{12}\right]\, e^{5-3x} + x^{12}\,\frac{d}{dx}\left[e^{5-3x}\right]$$
$$= 12x^{11}e^{5-3x} + x^{12}e^{5-3x}(-3)$$

The pieces

$$\frac{d}{dx}\left[x^{12}\right] = 12x^{11}$$

$$\frac{d}{dx}\left[e^{5-3x}\right] = e^{5-3x}\cdot(-3)$$ **Chain** outside $= e^{x} \xrightarrow{\frac{d}{dx}} e^{x}$, inside $= 5-3x \xrightarrow{\frac{d}{dx}} -3$

Summary

The Chain Rule

If $h(x) = f(g(x))$, then $h'(x) = f'\big(g(x)\big) \cdot g'(x)$: derivative of the **outside**, evaluated at the **inside**, times the derivative of the **inside**.

- Spot it by the word of: sin of $2x^2$, ln of $x^3 + x$, e to the $4x$, square root of $x + 2$, tan of a quotient.
- Write down *outside* and *inside*, differentiate each separately, then assemble: plug the inside into the derivative of the outside, and multiply by the derivative of the inside.
- The chain rule combines with the earlier rules: a product whose factor is a chain ($2x^4 \cos(3e^x)$, $x^{12}e^{5-3x}$), a quotient whose numerator is a chain ($\frac{\ln(x^3+30)}{8x}$), or a chain whose inside is a quotient (tan $\frac{\ln(x)}{2x-5}$).
- Rates chain too: $\dfrac{dL}{dt} = \dfrac{dL}{dw} \cdot \dfrac{dw}{dt}$ (the Tasmanian devil).