Section 3.1: Limits

The definition of a limit, and limits algebraically: back to the derivative

Definition of a limit

Definition (two-sided limit)

$\lim_{x \to a} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L as long as we choose x-values arbitrarily close to a **from either side**. This is called the two-sided limit.

Definition of a limit

Definition (two-sided limit)

$\lim_{x \to a} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L as long as we choose x-values arbitrarily close to a **from either side**. This is called the two-sided limit.

Definition (one-sided limits)

$\lim_{x \to a^-} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L as long as we choose x-values arbitrarily close to a **from the left**. This is called the left-hand limit.

Definition of a limit

Definition (two-sided limit)

$\lim_{x \to a} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L as long as we choose x-values arbitrarily close to a **from either side**. This is called the two-sided limit.

Definition (one-sided limits)

$\lim_{x \to a^-} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L as long as we choose x-values arbitrarily close to a **from the left**. This is called the left-hand limit.

$\lim_{x \to a^+} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L as long as we choose x-values arbitrarily close to a **from the right**. This is called the right-hand limit.

One-sided limits decide the two-sided limit

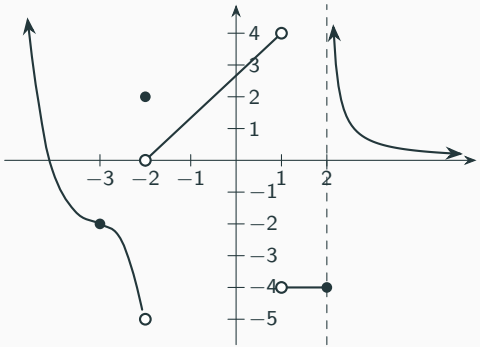
Theorem

Assume $f(x)$ is defined for all x near a except possibly at a. Then

$$\lim_{x \to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L.$$

Reading limits from a graph

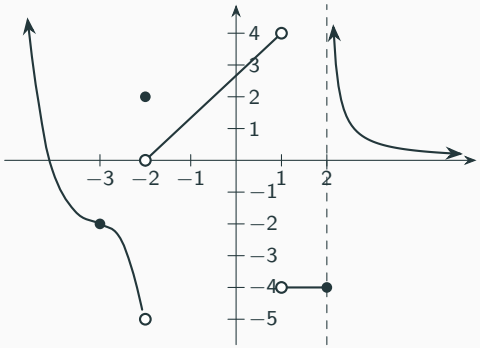
Example. Given the graph of $f(x)$, fill in the chart.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$
-3				
-2				
1				
2				

Reading limits from a graph

Example. Given the graph of $f(x)$, fill in the chart. *(We'll do the first row together.)*

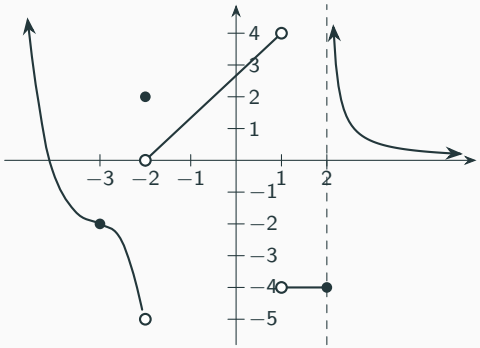


a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$
-3	-2	-2	-2	-2
-2				
1				
2				

Row 1: from both sides the graph heads to -2, and the filled dot gives $f(-3) = -2$.

Reading limits from a graph

Example. Given the graph of $f(x)$, fill in the chart. *(We'll do the first row together.)*



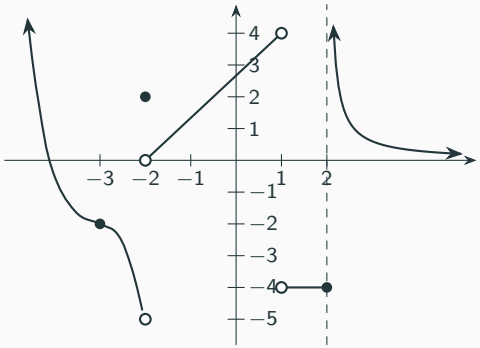
a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$
-3	-2	-2	-2	-2
-2				
1				
2				

Row 1: from both sides the graph heads to -2, and the filled dot gives $f(-3) = -2$.

Try it: now fill in the rest of the chart yourself.

Reading limits from a graph

Example. Given the graph of $f(x)$, fill in the chart. *(We'll do the first row together.)*



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$
-3	-2	-2	-2	-2
-2	-5	0	DNE	2
1	4	-4	DNE	undefined
2	-4	∞	DNE	-4

Row 1: from both sides the graph heads to -2, and the filled dot gives $f(-3) = -2$.

Try it: now fill in the rest of the chart yourself.

Limits algebraically (back to the derivative): Django's weight

A math professor found that, after introducing his dog Django to a new brand of food, Django's weight began to increase. After x weeks on the new food, Django's weight (in lbs) was approximately

$$w(x) = \sqrt{x} + 40, \qquad 0 \leq x \leq 6.$$

Try it

Find the instantaneous ROC of Django's weight after x weeks — that is, find $w'(x)$.

Limits algebraically (back to the derivative): Django's weight

A math professor found that, after introducing his dog Django to a new brand of food, Django's weight began to increase. After x weeks on the new food, Django's weight (in lbs) was approximately

$$w(x) = \sqrt{x} + 40, \qquad 0 \le x \le 6.$$

Try it

Find the instantaneous ROC of Django's weight after x weeks — that is, find $w'(x)$.

$$w'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} + \cancel{40} - \left(\sqrt{x} + \cancel{40}\right)}{h}$$

Django's weight: $\frac{0}{0}$ means do some algebra first

$$w'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{0}{0}$$ multiply by the conjugate

Django's weight: $\frac{0}{0}$ means do some algebra first

$$w'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{0}{0} \qquad \text{multiply by the conjugate}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

Django's weight: $\frac{0}{0}$ means do some algebra first

$$w'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{0}{0}$$ multiply by the conjugate

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

Django's weight: $\frac{0}{0}$ means do some algebra first

$$w'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{0}{0}$$ multiply by the conjugate

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Django's weight: $\frac{0}{0}$ means do some algebra first

$$w'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{0}{0}$$ multiply by the conjugate

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Django's weight: what the derivative tells us

$$w'(x) = \frac{1}{2\sqrt{x}} \qquad \text{(the instantaneous ROC of Django's weight after } x \text{ weeks)}$$

Django's weight: what the derivative tells us

$$w'(x) = \frac{1}{2\sqrt{x}} \qquad \text{(the instantaneous ROC of Django's weight after } x \text{ weeks)}$$

So for example, $x = 4$ (after 4 weeks)

$w(4) = \sqrt{4} + 40 = 42$ lbs — Django's weight.

$w'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$ — Django's weight is increasing at a rate of $\frac{1}{4}$ lb per week.

Rate of food intake during a meal

Studies on the eating behavior of humans revealed that for one subject, the cumulative intake of food during a meal can be described by

$$I(t) = 27 + 72t - 1.5t^2,$$

where t is the number of minutes since the meal began and $I(t)$ represents the amount of food (in g) that the person has eaten at time t.

Try it

Find the ROC of the intake of food for this particular person 1 min into a meal and 10 mins into a meal.

Rate of food intake during a meal

Studies on the eating behavior of humans revealed that for one subject, the cumulative intake of food during a meal can be described by

$$I(t) = 27 + 72t - 1.5t^2,$$

where t is the number of minutes since the meal began and $I(t)$ represents the amount of food (in g) that the person has eaten at time t.

Try it

Find the ROC of the intake of food for this particular person 1 min into a meal and 10 mins into a meal.

$$I'(t) = \lim_{h \to 0} \frac{I(t+h) - I(t)}{h}$$

Computing $I'(t)$

$$I'(t) = \lim_{h \to 0} \frac{27 + 72(t+h) - 1.5(t+h)^2 - \left(27 + 72t - 1.5t^2\right)}{h}$$

Computing $l'(t)$

$$l'(t) = \lim_{h \to 0} \frac{27 + 72(t+h) - 1.5(t+h)^2 - \left(27 + 72t - 1.5t^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{27} + \cancel{72t} + 72h - \cancel{1.5t^2} - 3th - 1.5h^2 - \cancel{27} - \cancel{72t} + \cancel{1.5t^2}}{h} = \frac{0}{0} \quad \text{factor}$$

Computing $l'(t)$

$$
\begin{aligned}
l'(t) &= \lim_{h \to 0} \frac{27 + 72(t+h) - 1.5(t+h)^2 - \left(27 + 72t - 1.5t^2\right)}{h} \\
&= \lim_{h \to 0} \frac{\cancel{27} + \cancel{72t} + 72h - \cancel{1.5t^2} - 3th - 1.5h^2 - \cancel{27} - \cancel{72t} + \cancel{1.5t^2}}{h} = \frac{0}{0} \quad \text{factor} \\
&= \lim_{h \to 0} \frac{\cancel{h}(72 - 3t - 1.5h)}{\cancel{h}}
\end{aligned}
$$

Computing $l'(t)$

$$
\begin{aligned}
l'(t) &= \lim_{h\to 0} \frac{27 + 72(t+h) - 1.5(t+h)^2 - \left(27 + 72t - 1.5t^2\right)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{27} + \cancel{72t} + 72h - \cancel{1.5t^2} - 3th - 1.5h^2 - \cancel{27} - \cancel{72t} + \cancel{1.5t^2}}{h} = \frac{0}{0} \quad \text{factor} \\
&= \lim_{h\to 0} \frac{\cancel{h}(72 - 3t - 1.5h)}{\cancel{h}} \\
&= 72 - 3t
\end{aligned}
$$

Interpreting $I'(t) = 72 - 3t$

When $t = 1$ and $t = 10$

$I'(1) = 72 - 3(1) = 69$ grams/min

$I'(10) = 72 - 3(10) = 42$ grams/min

Interpreting $I'(t) = 72 - 3t$

When $t = 1$ and $t = 10$

$I'(1) = 72 - 3(1) = 69$ grams/min $\qquad$ $I'(10) = 72 - 3(10) = 42$ grams/min

As the meal progresses, what is happening to the rate of intake of food?

Interpreting $I'(t) = 72 - 3t$

When $t = 1$ and $t = 10$

$I'(1) = 72 - 3(1) = 69$ grams/min $\qquad$ $I'(10) = 72 - 3(10) = 42$ grams/min

As the meal progresses, what is happening to the rate of intake of food?

It's **decreasing**!

Interpreting $I'(t) = 72 - 3t$

When $t = 1$ and $t = 10$

$I'(1) = 72 - 3(1) = 69$ grams/min $\qquad$ $I'(10) = 72 - 3(10) = 42$ grams/min

As the meal progresses, what is happening to the rate of intake of food?

It's **decreasing**!

When is the rate of intake zero?

Interpreting $I'(t) = 72 - 3t$

When $t = 1$ and $t = 10$

$I'(1) = 72 - 3(1) = 69$ grams/min $\qquad$ $I'(10) = 72 - 3(10) = 42$ grams/min

As the meal progresses, what is happening to the rate of intake of food?

It's **decreasing**!

When is the rate of intake zero?

$$72 - 3t = 0 \implies 72 = 3t \implies t = 24$$

So **24 mins after the meal began**.

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$

Example. Find an equation of the line tangent to $f(x) = \dfrac{-3}{x+1}$ at $x = 5$.

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$

Example. Find an equation of the line tangent to $f(x) = \dfrac{-3}{x+1}$ at $x = 5$.

$$\text{Slope} = f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$

Example. Find an equation of the line tangent to $f(x) = \dfrac{-3}{x+1}$ at $x = 5$.

$$\begin{aligned} \text{Slope} = f'(5) &= \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} \\ &= \lim_{h \to 0} \frac{\dfrac{-3}{(5+h)+1} - \left(\dfrac{-3}{6}\right)}{h} \end{aligned}$$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$

Example. Find an equation of the line tangent to $f(x) = \dfrac{-3}{x+1}$ at $x = 5$.

$$\begin{aligned}
\text{Slope} = f'(5) &= \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{-3}{(5+h)+1} - \left(\dfrac{-3}{6}\right)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{-3}{6+h} + \dfrac{3}{6}}{h} = \frac{0}{0} \qquad \text{get a common denominator}
\end{aligned}$$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$: a common denominator

$$f'(5) = \lim_{h \to 0} \frac{\frac{-3}{6+h} \cdot \frac{6}{6} + \frac{3}{6} \cdot \frac{6+h}{6+h}}{h}$$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$: a common denominator

$$f'(5) = \lim_{h \to 0} \frac{\frac{-3}{6+h} \cdot \frac{6}{6} + \frac{3}{6} \cdot \frac{6+h}{6+h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-18}{6(6+h)} + \frac{18+3h}{6(6+h)}}{h}$$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$: the slope

$$f'(5) = \lim_{h \to 0} \frac{\frac{-18 + 18 + 3h}{6(6+h)}}{h} = \lim_{h \to 0} \frac{3h}{6h(6+h)}$$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$: the slope

$$f'(5) = \lim_{h \to 0} \frac{\frac{-18 + 18 + 3h}{6(6+h)}}{h} = \lim_{h \to 0} \frac{3h}{6h(6+h)}$$

$$= \lim_{h \to 0} \frac{1}{2(6+h)} = \frac{1}{12}$$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$: the slope

$$f'(5) = \lim_{h \to 0} \frac{\frac{-18 + 18 + 3h}{6(6+h)}}{h} = \lim_{h \to 0} \frac{3h}{6h(6+h)}$$

$$= \lim_{h \to 0} \frac{1}{2(6+h)} = \frac{1}{12}$$

Point of tangency

$(5, f(5)) = \left(5, -\frac{1}{2}\right)$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$: the line

Equation of the tangent line

Slope $m = \frac{1}{12}$, point $\left(5, -\frac{1}{2}\right)$:

$y - y_1 = m(x - x_1)$

$y + \frac{1}{2} = \frac{1}{12}(x - 5)$

Tangent line to $f(x) = \frac{-3}{x+1}$ at $x = 5$: the line

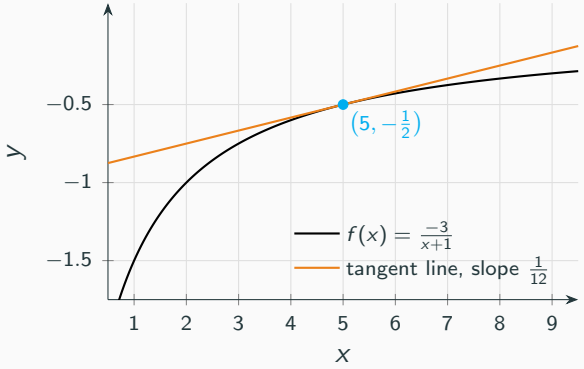
Equation of the tangent line

Slope $m = \frac{1}{12}$, point $\left(5, -\frac{1}{2}\right)$:

$y - y_1 = m(x - x_1)$

$y + \frac{1}{2} = \frac{1}{12}(x - 5)$

Check it: graph $f(x)$ and the tangent line on Desmos. The line just touches the curve at $\left(5, -\frac{1}{2}\right)$.



A derivative that does not exist: $f(x) = |x|$ at $x = 0$

Example. Given $f(x) = |x|$, find $f'(0)$.

A derivative that does not exist: $f(x) = |x|$ at $x = 0$

Example. Given $f(x) = |x|$, find $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{|0 + h| - |0|}{h}$$

A derivative that does not exist: $f(x) = |x|$ at $x = 0$

Example. Given $f(x) = |x|$, find $f'(0)$.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

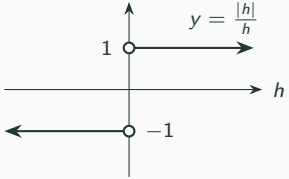
A derivative that does not exist: $f(x) = |x|$ at $x = 0$

Example. Given $f(x) = |x|$, find $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{|0 + h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$\frac{|h|}{h} = \begin{cases} -1 & \text{if } h < 0 \\ 1 & \text{if } h > 0 \end{cases}$$



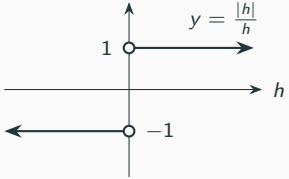
A derivative that does not exist: $f(x) = |x|$ at $x = 0$

Example. Given $f(x) = |x|$, find $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{|0 + h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$\frac{|h|}{h} = \begin{cases} -1 & \text{if } h < 0 \\ 1 & \text{if } h > 0 \end{cases}$$



Conclusion

$f'(0)$ **DNE**: the one-sided limits are -1 and 1 — NOT equal.

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$

Try it

Find the equation of the line tangent to $f(x) = -3\sqrt{x}$ at $x = 1$.

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$

Try it

Find the equation of the line tangent to $f(x) = -3\sqrt{x}$ at $x = 1$.

$$\text{Slope} = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{-3\sqrt{1+h} - (-3)}{h}$$

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$

Try it

Find the equation of the line tangent to $f(x) = -3\sqrt{x}$ at $x = 1$.

$$\text{Slope} = f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{-3\sqrt{1+h} - (-3)}{h}$$

$$= \lim_{h\to 0} \frac{-3\sqrt{1+h} + 3}{h} = \frac{0}{0}$$ multiply by the conjugate

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$

Try it

Find the equation of the line tangent to $f(x) = -3\sqrt{x}$ at $x = 1$.

$$\begin{aligned}
\text{Slope} = f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{-3\sqrt{1+h} - (-3)}{h} \\
&= \lim_{h\to 0} \frac{-3\sqrt{1+h} + 3}{h} = \frac{0}{0} \qquad \text{multiply by the conjugate} \\
&= \lim_{h\to 0} \frac{-3\sqrt{1+h} + 3}{h} \cdot \frac{3\sqrt{1+h} + 3}{3\sqrt{1+h} + 3}
\end{aligned}$$

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$, continued

$$f'(1) = \lim_{h \to 0} \frac{-9(1+h)+9}{h\left(3\sqrt{1+h}+3\right)}$$

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$, continued

$$f'(1) = \lim_{h \to 0} \frac{-9(1+h)+9}{h\left(3\sqrt{1+h}+3\right)}$$

$$= \lim_{h \to 0} \frac{-9-9h+9}{h\left(3\sqrt{1+h}+3\right)}$$

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$, continued

$$
\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{-9(1+h) + 9}{h\left(3\sqrt{1+h} + 3\right)} \\
&= \lim_{h \to 0} \frac{\cancel{-9} - 9\cancel{h} + \cancel{9}}{\cancel{h}\left(3\sqrt{1+h} + 3\right)} \\
&= \lim_{h \to 0} \frac{-9}{3\sqrt{1+h} + 3}
\end{aligned}
$$

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$, continued

$$
\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{-9(1+h)+9}{h\left(3\sqrt{1+h}+3\right)} \\
&= \lim_{h \to 0} \frac{-9-9h+9}{h\left(3\sqrt{1+h}+3\right)} \\
&= \lim_{h \to 0} \frac{-9}{3\sqrt{1+h}+3} \\
&= \frac{-9}{6} = -\frac{3}{2}
\end{aligned}
$$

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$, continued

$$
\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{-9(1+h)+9}{h\left(3\sqrt{1+h}+3\right)} \\
&= \lim_{h \to 0} \frac{-9-9h+9}{h\left(3\sqrt{1+h}+3\right)} \\
&= \lim_{h \to 0} \frac{-9}{3\sqrt{1+h}+3} \\
&= \frac{-9}{6} = -\frac{3}{2}
\end{aligned}
$$

Point of tangency and tangent line

Point of tangency: $(1, f(1)) = (1, -3)$, slope $m = -\frac{3}{2}$.

Tangent line to $f(x) = -3\sqrt{x}$ at $x = 1$, continued

$$
\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{-9(1+h)+9}{h\left(3\sqrt{1+h}+3\right)} \\
&= \lim_{h \to 0} \frac{\cancel{-9} - 9\cancel{h} + \cancel{9}}{\cancel{h}\left(3\sqrt{1+h}+3\right)} \\
&= \lim_{h \to 0} \frac{-9}{3\sqrt{1+h}+3} \\
&= \frac{-9}{6} = -\frac{3}{2}
\end{aligned}
$$

Point of tangency and tangent line

Point of tangency: $(1, f(1)) = (1, -3)$, slope $m = -\frac{3}{2}$. $\quad y + 3 = -\frac{3}{2}(x - 1)$

Summary

- $\lim_{x \to a} f(x) = L$ if and only if **both** one-sided limits equal L. On a graph, the left-hand limit, right-hand limit, two-sided limit, and $f(a)$ are four separate questions.
- The derivative by the limit definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, always starts out as $\frac{0}{0}$. Do algebra until h cancels, *then* let $h \to 0$:

If f has...	then...	Example
square roots	multiply by the conjugate	$\sqrt{x} + 40, \quad -3\sqrt{x}$
fractions	get a common denominator	$\frac{-3}{x+1}$
powers of x	expand, cancel, factor out h	$27 + 72t - 1.5t^2$

- Tangent line at $x = a$: $y - y_1 = m(x - x_1)$ with $m = f'(a)$ and $(x_1, y_1) = (a, f(a))$.
- $f'(a)$ can fail to exist: for $f(x) = |x|$, $f'(0)$ DNE since $|h|/h \to -1$ from the left but 1 from the right.