Section 3.1: Horizontal and Vertical Asymptotes

Limits at infinity, infinite limits, and what they look like on a graph

Example: a limit that is not a number

Example. Given $f(x) = \dfrac{3x+2}{2x+4}$, find $\lim\limits_{x \to -2^-} f(x)$ and $\lim\limits_{x \to -2^+} f(x)$ using a table of values.

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$x \to -2^-$	-2.1	-2.01	-2.001	-2.0001
$f(x)$	21.5	201.5	2001.5	20,001.5
$x \to -2^+$	-1.9999	-1.999	-1.99	-1.9
$f(x)$	$-19{,}998.5$	-1998.5	-198.5	-18.5

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From the left

$\lim\limits_{x\to -2^-} f(x) = \infty$: as $x \to -2$ from the left, $f(x)$ becomes larger and larger without bound.

Why does $f(x)$ blow up?

Why?

Because as x approaches -2, the **denominator** $2x + 4$ approaches zero, while the **numerator** $3x + 2$ approaches -4 — and -4 divided by a smaller and smaller number becomes larger and larger.

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$\lim_{x \to -2^+} f(x) = -\infty$: as x approaches -2 from the right, $f(x)$ becomes larger and larger in the **negative** direction without bound.

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$\lim_{x \to -2^+} f(x) = -\infty$: as x approaches -2 from the right, $f(x)$ becomes larger and larger in the **negative** direction without bound.

(From the right the denominator is a tiny *positive* number, so -4 divided by it is a huge negative number.)

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x	10	100	1000	10,000	100,000
$f(x)$	1.33	1.48	1.498	1.4998	1.49998

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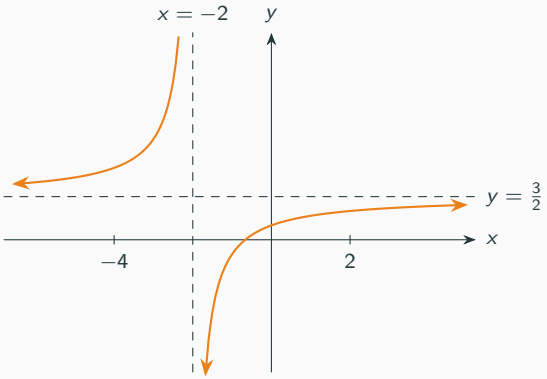
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$f(x)$	1.75	1.52	1.502	1.5002	1.50002

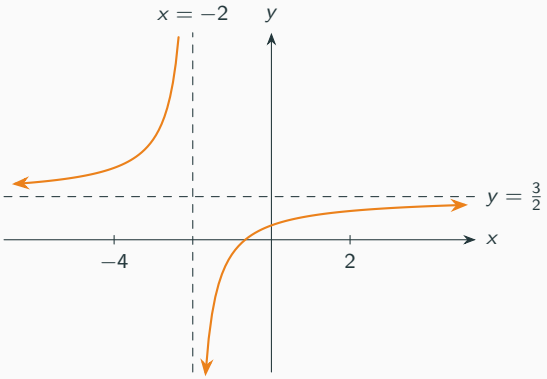
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Graphically



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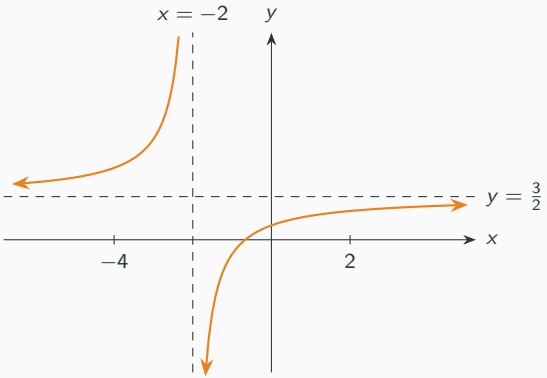
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The dashed horizontal line $y = \frac{3}{2}$ is the value $f(x)$ levels off to as $x \to \pm\infty$.

Definition: vertical asymptote

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$x = -2$ is a VA since $\lim_{x \to -2^-} f(x) = \infty$ and $\lim_{x \to -2^+} f(x) = -\infty$.

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(Either one-sided infinite limit is enough for a vertical asymptote.)

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Note

A function can have **infinitely many** VAs (e.g. $f(x) = \tan(x)$), but **no more than 2** HAs — one for $x \to \infty$ and one for $x \to -\infty$, at most.

The key limit for finding horizontal asymptotes

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Strategy for rational functions $\frac{p(x)}{q(x)}$

Factor the **highest power of x in $q(x)$** (the denominator) from the numerator *and* the denominator. The leftover fractions $\frac{c}{x^n}$ all go to 0.

Example: find all HAs of $f(x) = \frac{5x^2-4x^3}{3x^2+2x-1}$, as $x \to \infty$

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$$\lim_{x\to\infty} \frac{5x^2 - 4x^3}{3x^2 + 2x - 1} = \lim_{x\to\infty} \frac{x^2(5 - 4x)}{x^2\left(3 + \frac{2}{x} - \frac{1}{x^2}\right)}$$

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$$= \frac{5 - 4(\infty)}{3} = -\infty \qquad \Longleftarrow \text{ NO HA from this side}$$

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Conclusion

This function has **NO** horizontal asymptotes.

Try it: find all HAs of $f(x) = \frac{3x^2-5x+1}{4x^2-7}$

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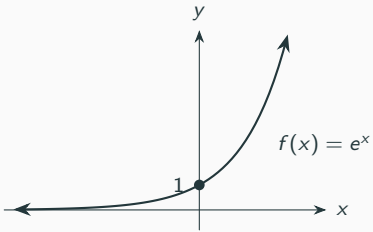
Conclusion

Both limits are $\frac{3}{4}$, so $y = \frac{3}{4}$ is the **ONLY** HA.

Functions involving e^x

The two limits to know

$$\lim_{x \to \infty} e^x = e^{\infty} = \infty$$

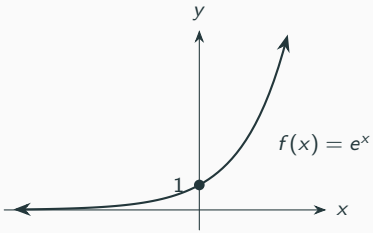


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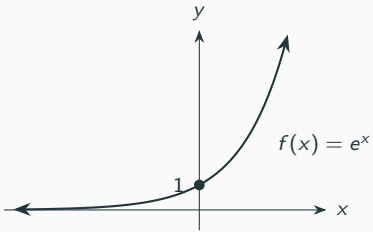
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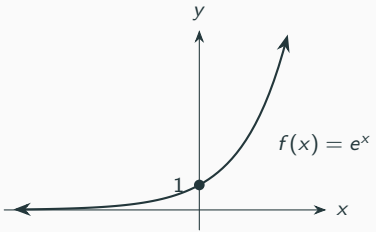
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Building blocks ($k > 0$)

	$x \to \infty$	$x \to -\infty$
e^{kx} (like e^x)	∞	0
$e^{-kx} = \frac{1}{e^{kx}}$	0	∞

A strategy for HAs with exponentials

Strategy (do $x \to \infty$ and $x \to -\infty$ separately)

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3. If **some** exponential $\to \infty$: factor the **biggest exponential in the denominator** out of the top *and* the bottom, and cancel it — just like the highest power of x for rational functions. Every leftover $\frac{c}{e^{kx}} \to 0$.

Which exponential is the biggest as $x \to \infty$?

The one with the largest number in front of x. At $x = 5$: $e^x \approx 148$, $e^{2x} \approx 22{,}026$, $e^{3x} \approx 3{,}269{,}017$.

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4. A number L: $y = L$ is an HA. $\pm\infty$: no HA on that side.

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$$= \lim_{x\to\infty} \frac{\underbrace{\frac{2}{e^{2x}}}_{\to 0} - \underbrace{\frac{15}{e^{3x}}}_{\to 0}}{5 + \underbrace{\frac{8}{e^{3x}}}_{\to 0}} = \frac{0 - 0}{5 + 0} = 0 \qquad \Longleftarrow\ y = 0 \text{ is an HA}$$

Example (a), continued: as $x \to -\infty$

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Step 2. Every exponential $\to 0$, so replace each one by 0:

$$\lim_{x \to -\infty} \frac{2\underbrace{e^x}_{\to 0} - 15}{5\underbrace{e^{3x}}_{\to 0} + 8} = \frac{2(0) - 15}{5(0) + 8} = -\frac{15}{8}$$

So $y = -\frac{15}{8}$ is also an HA.

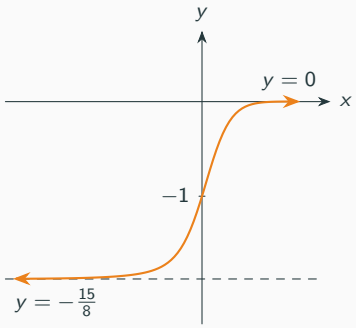
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Conclusion

Two HAs: $y = 0$ (as $x \to \infty$) and $y = -\frac{15}{8}$ (as $x \to -\infty$).

Try it: all HAs of $f(x) = \frac{3+7e^{2x}}{1-e^x}$

Try it (b). Calculate all HAs of $f(x) = \dfrac{3 + 7e^{2x}}{1 - e^x}$. Follow the strategy, one side at a time.

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$$= \lim_{x\to\infty} \frac{\underbrace{\frac{3}{e^x}}_{\to 0} + \underbrace{7e^x}_{\to\infty}}{\underbrace{\frac{1}{e^x}}_{\to 0} - 1} = \frac{0+\infty}{0-1} = -\infty \qquad \Longleftarrow \text{ no HA on this side}$$

Try it (b), continued: as $x \to -\infty$

As $x \to -\infty$. Step 1: $e^{2x} \to 0$ and $e^{x} \to 0$.

Try it (b), continued: as $x \to -\infty$

As $x \to -\infty$. Step 1: $e^{2x} \to 0$ and $e^x \to 0$.

Step 2: every exponential $\to 0$, so replace each one by 0:

$$\lim_{x \to -\infty} \frac{3 + 7\underbrace{e^{2x}}_{\to 0}}{1 - \underbrace{e^x}_{\to 0}} = \frac{3 + 7(0)}{1 - 0} = 3$$

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Conclusion

$y = 3$ is the **only** HA (as $x \to -\infty$). As $x \to \infty$, $f(x) \to -\infty$, so there is no HA on that side.

Logistic growth of a population

The logistic curve describes the growth of a population over time, where the rate of growth depends on the population size. The logistic equation is one model for *density-dependent* growth and models the growth of a population whose reproductive rate decreases as the population size increases because of competition among organisms for territory or resources.

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The logistic curve

If $N(t)$ denotes the size of the population at time t, then

$$N(t) = \frac{k}{1 + ae^{-rt}} \qquad \text{for } t \geq 0,$$

where k and r are positive numbers that describe the population dynamics and $a > -1$ is a coefficient that determines the initial population size.

What is the long-term behavior of the population?

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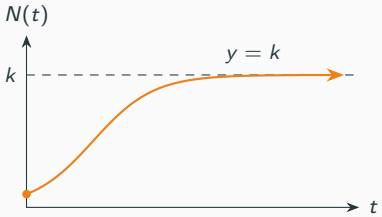
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Carrying capacity

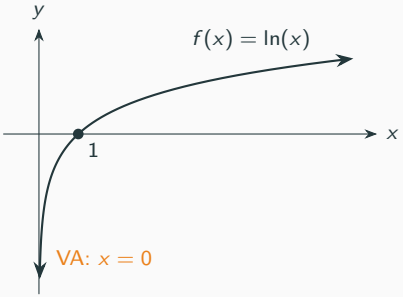
k is the **carrying capacity**: the maximum number of organisms that the habitat can sustain. On the graph, $y = k$ is a horizontal asymptote of $N(t)$.

Functions involving $\ln(x)$

The two limits to know

$\lim_{x\to\infty} \ln(x) = \infty$ (slowly: $\ln(1{,}000{,}000) \approx 13.8$)

$\lim_{x\to 0^+} \ln(x) = -\infty$ ($\ln(x)$ is only defined for $x > 0$)



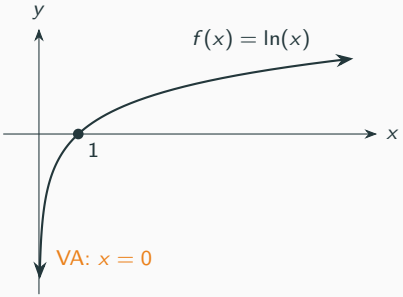
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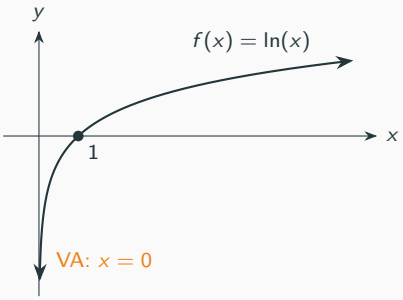
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So $\ln(x)$ has the VA $x = 0$ (the y-axis) and **no** HA.

Strategy for $\ln($**inside**$)$

VA where the inside $\to 0^+$: solve inside $= 0$.
e.g. $\ln(x-2) \to -\infty$ as $x \to 2^+$: VA $x = 2$.
No HA: $\ln(x-2) \to \infty$ as $x \to \infty$.



Summary

Asymptotes

$x = a$ is a **vertical asymptote** if $f(x) \to \pm\infty$ as $x \to a$ from either side (or both); $y = L$ is a **horizontal asymptote** if $f(x) \to L$ as $x \to \infty$ or as $x \to -\infty$.

- At most 2 HAs (check $x \to \pm\infty$ separately), but any number of VAs (e.g. $\tan x$).
- Rational functions: factor the highest power of x in the denominator out of the top and bottom; $\frac{c}{x^n} \to 0$.
- Exponentials: if every exponential $\to 0$, replace each by 0; if some $\to \infty$, factor the biggest exponential in the denominator out of the top and bottom.
- $\ln(x) \to -\infty$ as $x \to 0^+$ (VA $x = 0$), and $\ln(x) \to \infty$ as $x \to \infty$ (no HA).
- Logistic growth $N(t) = \frac{k}{1+ae^{-rt}}$: $e^{-rt} \to 0$, so $N(t) \to k$, the carrying capacity.