Section 3.3: Rates of Change

From average rates to the derivative

Average rate of change: the flu in College Station

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Try it

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What does the 6 mean?

Between days 1 and 4, the percent of the population infected increased by an **average of 6% per day**. (An average rate is a rate over an *interval*.)

An average rate of change is a slope

Note

The average ROC over $[1, 4]$ is exactly the **slope of the line** through $(1, p(1)) = (1, 2)$ and $(4, p(4)) = (4, 20)$:

$$m = \frac{20 - 2}{4 - 1} = 6.$$

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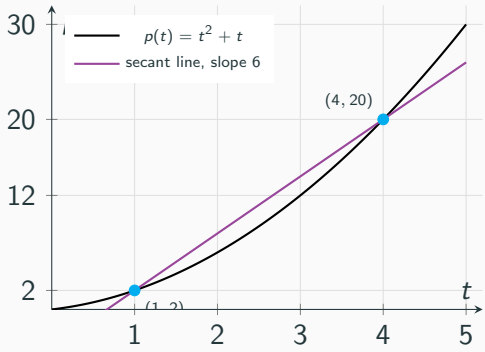
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Definition

Any line connecting two points on a curve is called a **secant line**.



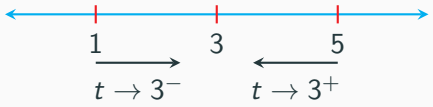
Instantaneous rate of change: zooming in on $t = 3$

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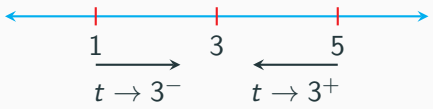
Idea: we only know how to compute rates over intervals, so compute the average ROC over smaller and smaller intervals $[3, t]$ and watch what happens.



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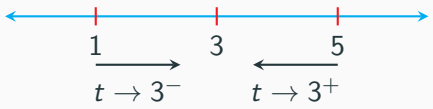


Interval $[3, t]$	$\frac{p(t) - p(3)}{t - 3}$, $p(3) = 12$	Avg. ROC
$[3, 5]$	$\frac{5^2 + 5 - 12}{5 - 3}$	9
$[3, 4]$	$\frac{4^2 + 4 - 12}{4 - 3}$	8
$[3, 3.5]$	$\frac{(3.5)^2 + 3.5 - 12}{3.5 - 3}$	7.5
$[3, 3.1]$	$\frac{(3.1)^2 + 3.1 - 12}{3.1 - 3}$	7.1
$[3, 3.01]$	$\frac{(3.01)^2 + 3.01 - 12}{3.01 - 3}$	7.01
$[3, 3.001]$	$\frac{(3.001)^2 + 3.001 - 12}{3.001 - 3}$	7.001

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A good approximation for the instantaneous ROC at $t = 3$ is 7.

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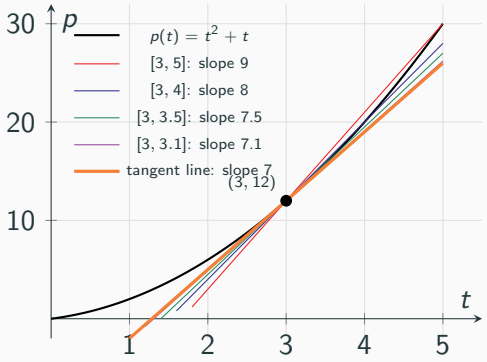
"As t gets arbitrarily close to 3 from the right, the average ROC gets arbitrarily close to 7."

Approaching from the left, with intervals $[t, 3]$, gives the same value:

$$\lim_{t \to 3^-} \frac{p(3) - p(t)}{3 - t} = 7.$$

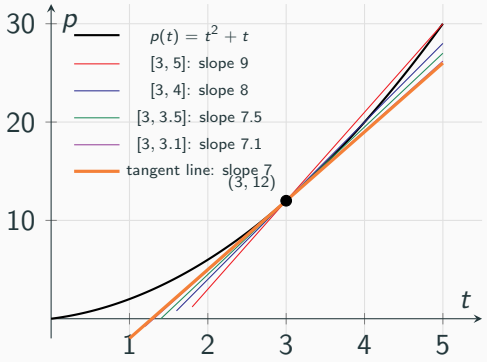
(Fill in the $[t, 3]$ table on your own — you should reach the same conclusion.)

What the secant lines are doing: the tangent line



Observations as $t \to 3^+$:

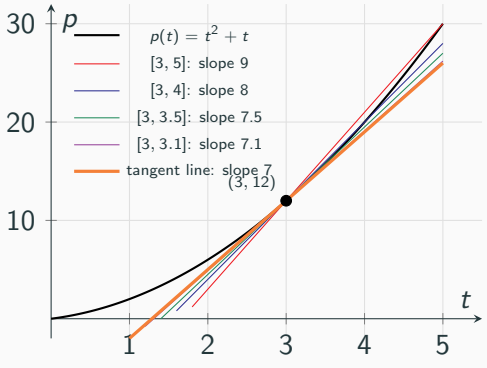
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Observations as $t \to 3^+$:

1. The secant lines approach one unique line, the tangent line at $(3, 12)$. It shows the direction the curve is heading at that point.

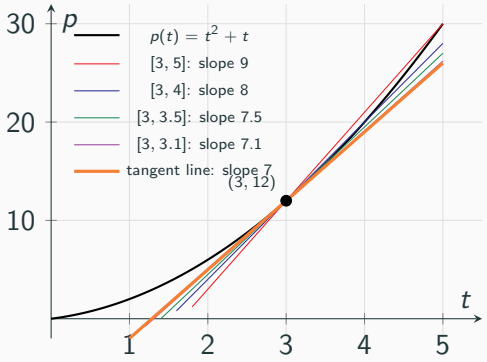
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Equation of the tangent line at $t = 3$

Point $(3, 12)$, slope $m = 7$:

$$y - 12 = 7(t - 3)$$

Making it general: the derivative

Definition (instantaneous rate of change)

The instantaneous ROC of $f(x)$ at $x = a$ is the limit of the average ROC,

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} \quad = \quad \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided the limit exists.

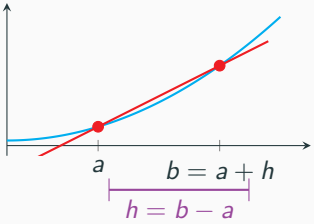
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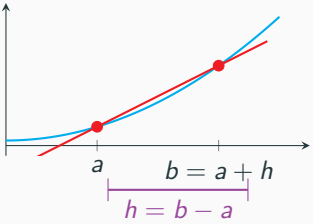
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Definition (derivative)

The **derivative of** f **at** x ("f prime of x") is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{provided the limit exists.}$$

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Note

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Check with the flu example: $a = 3$, $p(3) = 12$, $p'(3) = 7$ gives $y - 12 = 7(t - 3)$, the same line as before.

Summary

	Average ROC	**Instantaneous ROC**
Where	over an interval $[a, b]$	at a single point $x = a$
Formula	$\dfrac{f(b) - f(a)}{b - a}$	$f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
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Big idea: an instantaneous rate is a **limit** of average rates as the interval shrinks to a point.