Section 11.1: Solutions of Elementary and Separable Differential Equations

Finding a function from an equation for its rate of change

Example 1: a growing flock of birds

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Initial value problem strategy (7.1)

(1) antidifferentiate the rate; (2) find C from the known value; (3) write the function; (4) answer the question.

Example 1, solved

Step 1 (antidifferentiate), using $\int e^{kt}\,dt = \frac{1}{k}\,e^{kt} + C$:

$$P(t) = \int 20e^{0.05t}\,dt = \frac{20}{0.05}\,e^{0.05t} + C = 400e^{0.05t} + C$$

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Step 3: the function

$P(t) = 400e^{0.05t} - 380$ birds after t years

Check: $P'(t) = 400(0.05)\,e^{0.05t} = 20e^{0.05t}$ and $P(0) = 400 - 380 = 20$.

Differential equations

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Example 1 was an initial value problem, with $P(0) = 20$. Its right side had only t in it, so integrating was enough. Next: a rate that depends on the unknown function itself.

Example 2: Tylenol in the bloodstream

Example 2. After a dose of Tylenol is taken, the drug is absorbed into the bloodstream. Assume that the amount of Tylenol in the bloodstream changes at a rate proportional to the amount currently present. (This is a common model for first-order drug kinetics.) Let

- $A(t)$ = the amount of Tylenol in the bloodstream (mg) at time t (hours),
- k = the elimination rate constant. Since Tylenol is being eliminated from the body, $k < 0$.

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(a) What is the differential equation? (b) Find the general solution for the amount of Tylenol in the bloodstream.

Try it first: write down the differential equation.

Step 1: write the differential equation

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Strategy, Step 1: the differential equation

Translate the words into an equation for the rate of change. "Proportional to" means "a constant times".

Step 2: separate the variables

Treat $\frac{dA}{dt}$ like a fraction. Divide both sides by A and multiply both sides by dt:

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Strategy, Step 2: separate the variables

Put everything with the unknown function on the side with its differential (dA), and everything with t on the side with dt.

Step 3: integrate both sides

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Strategy, Step 3: integrate both sides

Integrate the left side with respect to the unknown function and the right side with respect to t, and write one $+\,C$.

Step 4: solve for A

Exponentiate both sides to undo the logarithm:

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e^{C} is a positive constant; call it M. And A is an amount of Tylenol, so it cannot be negative: $|A| = A$. Therefore

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Strategy, Step 4: solve for the function

Undo the logarithm by exponentiating, write $e^{kt+C} = e^{kt}e^{C}$, and rename the constant e^{C} as M.

Step 5: the constant

At $t = 0$: $A(0) = M\,e^{k(0)} = M$. So M is the amount of Tylenol in the bloodstream at time 0.

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Check: $\frac{d}{dt}\left[M\,e^{kt}\right] = k\,M\,e^{kt} = kA$, so $A(t)$ satisfies the equation.

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Check: $\frac{d}{dt}\left[M\,e^{kt}\right] = k\,M\,e^{kt} = kA$, so $A(t)$ satisfies the equation.

Strategy, Step 5: use the initial value

Find the constant from the initial value. When no value is given, as here, the constant stays in the **general solution**; here it is $M = A(0)$.

The strategy we just built

Solving a differential equation by separating the variables

1. **Write the differential equation** from the words.
2. **Separate the variables:** the unknown function with its differential, t with dt.
3. **Integrate both sides**, with one $+ C$.
4. **Solve for the function:** exponentiate, and rename e^C as a new constant.
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Exponential growth and decay

The solution to $\frac{dy}{dt} = ky$ is the exponential growth/decay function $y = M e^{kt}$, where k is the growth constant and M is the amount present at $t = 0$.

Limited growth

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Assume a **maximum population** of size N, and that the rate of growth is proportional to how close the population is to that maximum. **Try it first:** write down the differential equation.

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The limited growth model

$$\frac{dy}{dt} = k(N - y)$$

Here $N - y$ is how far the population is below the maximum. The solution is the **limited growth function**.

Example 3: mountain goats

Example 3. A certain nature preserve can support no more than 4000 mountain goats. Assume that the rate of growth is proportional to how close the population is to this maximum, with a growth rate of 20%. There are currently 1000 goats in the area.

(a) Solve the general limited growth differential equation, and then write a function describing the goat population at time t.

(b) What will the population be in 5 years?

(c) What happens as $t \to \infty$?

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(a) Solve the general limited growth differential equation, and then write a function describing the goat population at time t.

(b) What will the population be in 5 years?

(c) What happens as $t \to \infty$?

Try it first: use the five steps on $\frac{dy}{dt} = k(N - y)$, with the initial value $y(0) = y_0$.

Example 3(a): Steps 2 and 3

Step 2 (separate): $\dfrac{dy}{dt} = k(N - y) \implies \dfrac{1}{N - y}\, dy = k\, dt$

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The left side is a substitution (Section 7.2): with $u = N - y$, $du = -dy$, so $\int \frac{1}{N-y}\,dy = -\int \frac{1}{u}\,du = -\ln|u| = -\ln|N-y|$ (plus a constant).

Example 3(a): Steps 4 and 5

Step 4 (solve for y): multiply by -1, then exponentiate. The population is below the maximum, so $N - y > 0$:

$$\ln(N - y) = -kt - C \implies N - y = e^{-kt}\, e^{-C} = M\, e^{-kt} \implies y = N - M\, e^{-kt}$$

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Step 5 (initial value): $y(0) = y_0$ gives $N - M\, e^{-k(0)} = y_0$, so $N - M = y_0$ and $M = N - y_0$.

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Step 5 (initial value): $y(0) = y_0$ gives $N - M\, e^{-k(0)} = y_0$, so $N - M = y_0$ and $M = N - y_0$.

The limited growth function

$$y = N - (N - y_0)\, e^{-kt}$$

Example 3(a): the goat population

Put $N = 4000$, $y_0 = 1000$ and $k = 0.2$ (a growth rate of 20%) into the limited growth function $y = N - (N - y_0)\, e^{-kt}$:

$$y = 4000 - (4000 - 1000)\, e^{-0.2t}$$

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Answer to (a)

$y = 4000 - 3000e^{-0.2t}$ goats after t years

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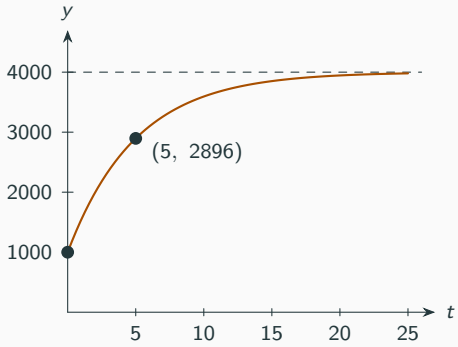
Check: $y(0) = 4000 - 3000 = 1000$, the current population.

Example 3(b) and (c)

(b) Put $t = 5$, so that $e^{-0.2(5)} = e^{-1}$:

$$y(5) = 4000 - 3000e^{-1} \approx 2896$$

In 5 years there will be about **2896 goats**.



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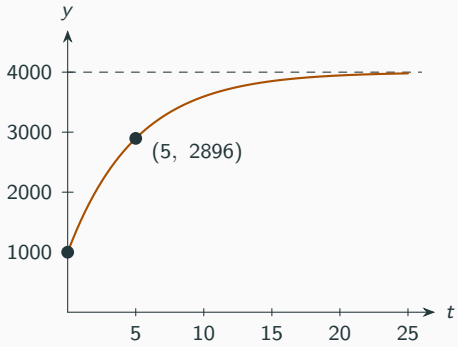
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(c) As $t \to \infty$, $e^{-0.2t} \to 0$, so

$$y(t) \to 4000,$$

the carrying capacity of the preserve (recall Section 3.1).



Summary

Separating the variables

(1) write the differential equation; (2) separate the variables; (3) integrate both sides, with one $+ C$; (4) solve for the function (exponentiate, rename e^C); (5) use the initial value to find the constant.

- Birds: $\frac{dP}{dt} = 20e^{0.05t}$, $P(0) = 20 \;\Rightarrow\; P(t) = 400e^{0.05t} - 380$ (integrate directly).
- Exponential growth or decay: $\frac{dy}{dt} = ky \;\Rightarrow\; y = M\,e^{kt}$, with $M = y(0)$. Tylenol: $A(t) = M\,e^{kt}$ with $k < 0$.
- Limited growth: $\frac{dy}{dt} = k(N - y) \;\Rightarrow\; y = N - (N - y_0)\,e^{-kt}$, which approaches N. Goats: $y = 4000 - 3000e^{-0.2t}$, about 2896 after 5 years.

Next time: back to the logistic growth problem from Section 3.1 (horizontal and vertical asymptotes).