# Section 11.1 (Part 2): Logistic Growth

Growth that levels off at a carrying capacity

# Three growth models

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We solved the first two in Section 11.1. Today: the logistic model.

## Example 1: fruit flies in a lab container

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- $P(t)$ = the number of fruit flies $t$ days after the experiment begins,
- $N$ = the carrying capacity, and $k > 0$ = the population growth constant.

**The logistic equation**

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

## What the logistic equation says

**When $P$ is small:** $1 - \frac{P}{N} \approx 1$, so $\frac{dP}{dt} \approx kP$.

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**The idea of the model**

The logistic equation is the unlimited growth equation $\frac{dP}{dt} = kP$ with the extra factor $\left(1 - \frac{P}{N}\right)$, which accounts for the limiting environmental factors when $P$ nears $N$.

# Solving the logistic equation: Steps 1 and 2

**Separating the variables (Section 11.1)**

(1) the differential equation; (2) separate the variables; (3) integrate both sides; (4) solve for the function; (5) use the initial value.

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**Step 1:** $\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$

**Step 2 (separate):** since $1 - \frac{P}{N} = \frac{N-P}{N}$,

$$\frac{dP}{P\left(1 - \frac{P}{N}\right)} = k\,dt \qquad \Longrightarrow \qquad \frac{N}{(N-P)P}\,dP = k\,dt$$

## Step 3: integrate both sides

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so $\ln\left|\dfrac{P}{N-P}\right| = kt + C$. (As in 11.1, $\int \frac{1}{N-P}\,dP = -\ln|N-P|$, by the substitution $u = N - P$.)

## Step 4: solve for $P$

The population stays between 0 and $N$, so $\frac{P}{N-P} > 0$. Exponentiate, and call the constant $e^C = M$:

$$\frac{P}{N-P} = e^{kt}e^C = M\,e^{kt}$$

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Multiply by $N - P$, then collect the terms with $P$:

$$P = N\,M\,e^{kt} - P\,M\,e^{kt} \implies P\left(1 + M\,e^{kt}\right) = N\,M\,e^{kt} \implies P = \frac{N\,M\,e^{kt}}{1 + M\,e^{kt}}$$

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Divide the top and the bottom by $M\,e^{kt}$, and call the constant $b = \frac{1}{M}$:

$$P = \frac{N}{\frac{1}{M\,e^{kt}} + 1} = \frac{N}{1 + b\,e^{-kt}}$$

## Step 5: use the initial value

Let $P(0) = P_0$, the initial population. Then

$$P_0 = \frac{N}{1 + b\,e^{-k(0)}} = \frac{N}{1+b} \implies (1+b)P_0 = N \implies b = \frac{N - P_0}{P_0}$$

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**The logistic function**

$$P(t) = \frac{N}{1 + \left(\dfrac{N - P_0}{P_0}\right) e^{-kt}}$$

$N$ = the carrying capacity, $P_0$ = the initial population, $k$ = the growth constant

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Check: $P(0) = \dfrac{N}{1 + \frac{N-P_0}{P_0}} = \dfrac{N\,P_0}{P_0 + N - P_0} = P_0.$

## Example 2: 30 fruit flies

**Example 2.** A biologist begins an experiment with 30 fruit flies. The carrying capacity is 500 fruit flies, and the growth constant is $k = 0.20$ per day.

(a) Write the logistic model.

(b) What happens as $t \to \infty$? What does this mean?

(c) Determine the fruit fly population after 10 days.

(d) How many days will it take for the fruit fly population to reach 250?

(e) At what population $P$ is the fruit fly population increasing most rapidly? What is the maximum growth rate?

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**Try it first:** answer each part before we go through them.

# Example 2(a) and (b)

**(a)** $N = 500$, $P_0 = 30$, $k = 0.20$, and $\dfrac{N - P_0}{P_0} = \dfrac{500 - 30}{30} = \dfrac{47}{3}$:

$$P(t) = \frac{500}{1 + \frac{47}{3}\, e^{-0.20t}}$$

## Example 2(a) and (b)

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**(b)** As $t \to \infty$, $e^{-0.20t} \to 0$, so

$$\lim_{t \to \infty} P(t) = \frac{500}{1 + 0} = 500.$$

The fruit fly population approaches the carrying capacity of 500 flies.

# Example 2(c)

**(c)** Put $t = 10$ into the model:

$$P(10) = \frac{500}{1 + \frac{47}{3}\, e^{-0.20(10)}} = \frac{500}{1 + \frac{47}{3}\, e^{-2}} \approx 160$$

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**Answer to (c)**

There are about **160 fruit flies** after 10 days.

# Example 2(d)

**(d)** Solve $P(t) = 250$:

$$250 = \frac{500}{1 + \frac{47}{3}\, e^{-0.20t}} \implies 250\left(1 + \frac{47}{3}\, e^{-0.20t}\right) = 500 \implies 1 + \frac{47}{3}\, e^{-0.20t} = 2$$

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**Strategy: when does the population reach a given size?**

Set $P(t)$ equal to that size, clear the fraction, isolate the exponential, take ln of both sides, and solve for $t$.

# Example 2(e): when is the population growing fastest?

The growth rate is a function of $P$:

$$\frac{dP}{dt} = 0.20\left(1 - \frac{P}{500}\right)P = 0.20P - 0.0004P^2$$

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This derivative is positive for $P < 250$ and negative for $P > 250$: the growth rate increases, then decreases. $P = 250$ is the only critical point in $(0, 500)$, so it gives the absolute max.

## Example 2(e): the maximum growth rate

At $P = 250$: $\dfrac{dP}{dt} = 0.20\left(1 - \dfrac{250}{500}\right)(250) = 25.$

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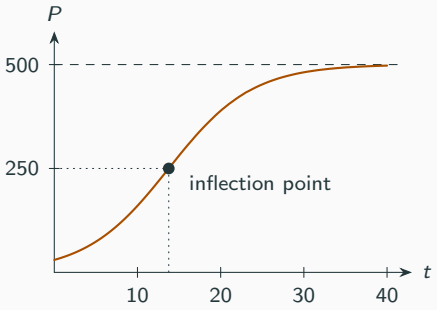
**Strategy: when is the population growing fastest?**

Treat the growth rate $\frac{dP}{dt}$ as a function of $P$, and find its absolute max with the methods of Section 6.1.

# The inflection point (the last example of Sections 5.1–5.3)

For logistic growth, the **inflection point** is always at half the carrying capacity.

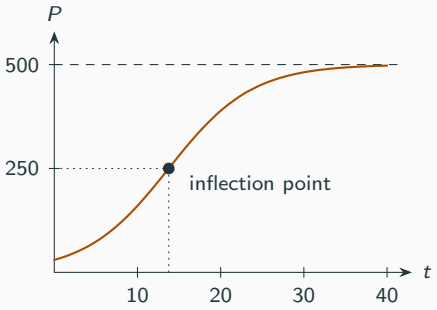
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**Three ways to say the same thing**

half the carrying capacity ($P = 250$) $\iff$ the inflection point of $P(t)$ ($t \approx 13.8$) $\iff$ the maximum of $P'(t)$ ($P'(t) = 25$ flies per day)

# Summary

- Unlimited growth $\frac{dy}{dt} = ky$; limited growth $\frac{dy}{dt} = k(N - y)$; logistic growth $\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right) y$.
- Separating the variables (with partial fractions) solves the logistic equation: $P(t) = \dfrac{N}{1 + \left(\frac{N - P_0}{P_0}\right) e^{-kt}}$, and $P(t) \to N$ as $t \to \infty$.
- Fruit flies: $P(t) = \dfrac{500}{1 + \frac{47}{3}\, e^{-0.20t}}$; about 160 flies after 10 days; 250 flies after about 13.8 days.
- The population grows fastest at half the carrying capacity: the inflection point of $P(t)$, where the growth rate has its maximum (25 flies per day here).