Midterm Review

Key definitions and practice problems: Sections 3.1–3.4, 4.1–4.3, 5.1–5.3, and 6.5

Key facts: limits, asymptotes, and continuity (3.1, 3.2)

Limits

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Continuity

f is continuous at $x = c$ if (1) $f(c)$ is defined, (2) $\lim_{x \to c} f(x)$ exists, and (3) $\lim_{x \to c} f(x) = f(c)$.

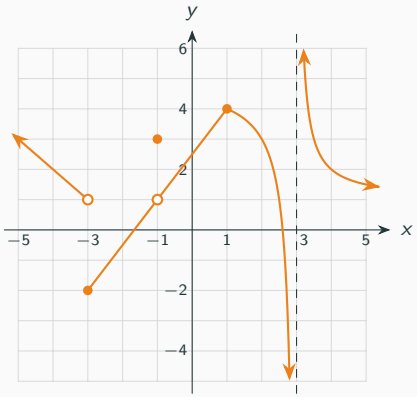
Asymptotes

$x = a$ is a vertical asymptote if $f(x) \to \infty$ or $-\infty$ as $x \to a^-$ or as $x \to a^+$.

$y = L$ is a horizontal asymptote if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Key limits: $\frac{1}{x^n} \to 0$ as $x \to \pm\infty$ $(n > 0)$; $e^x \to \infty$ as $x \to \infty$; $e^x \to 0$ as $x \to -\infty$.

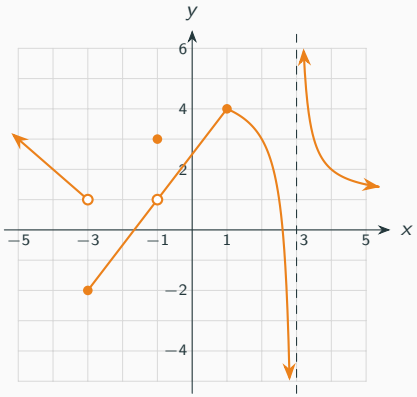
Exercise 1: limits, continuity, and f' from a graph



Exercise. (a) Use the graph of $y = f(x)$ to fill in the chart (DNE if a limit does not exist). (b) Where in $(-5, 5)$ does $f'(x)$ fail to exist? Why?

a	$\lim_{x \to a^-} f(x)$	$\lim_{x \to a^+} f(x)$	$\lim_{x \to a} f(x)$	$f(a)$	cont.?
-3					
-1					
1					
3					

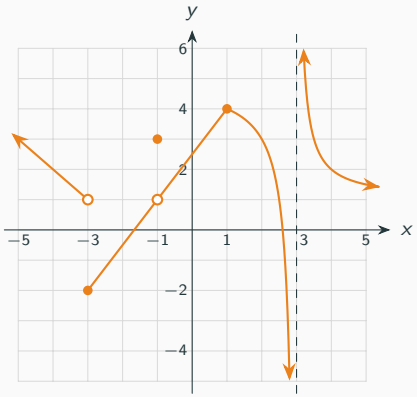
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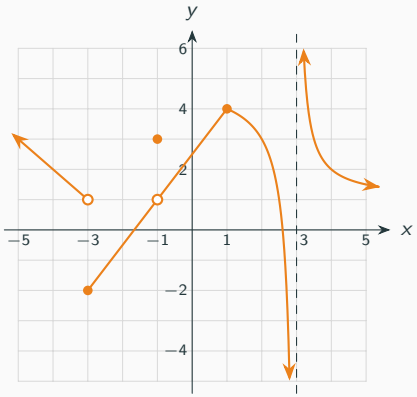
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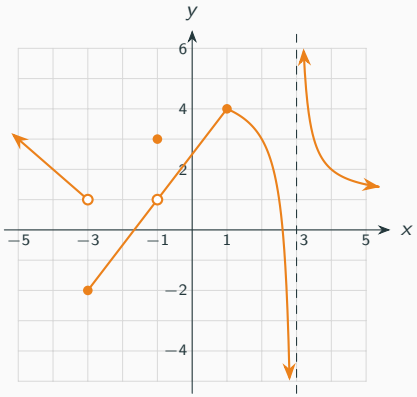
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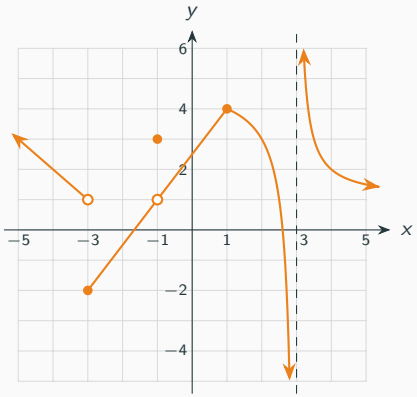
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(b) $f'(x)$ DNE at $x = -3, -1, 1, 3$

At -3, -1, 3: not continuous. At 1: a **corner**.

Exercise 2: asymptotes of a rational function

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Let $f(x) = \dfrac{2x^2 - 2x - 4}{x^2 - 4}$. (a) Find all horizontal asymptotes.
(b) Find all vertical asymptotes, and both one-sided limits at each one.

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$$\lim_{x \to \pm\infty} \frac{\cancel{x^2}\left(2 - \frac{2}{x} - \frac{4}{x^2}\right)}{\cancel{x^2}\left(1 - \frac{4}{x^2}\right)} = \frac{2 - 0 - 0}{1 - 0} = 2$$

So $y = 2$ is the only HA.

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(b) Factor: $f(x) = \dfrac{2(x-2)(x+1)}{(x-2)(x+2)}$

$x = 2$: $x - 2$ cancels; $\lim_{x \to 2} \frac{2(x+1)}{x+2} = \frac{3}{2}$:
a number, so a **hole**, **NOT** a VA.

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$\lim_{x \to -2^-} f(x) = \infty \quad \left(\frac{-2}{\text{tiny negative}}\right)$

$\lim_{x \to -2^+} f(x) = -\infty \quad \left(\frac{-2}{\text{tiny positive}}\right)$

Exercise 3: horizontal asymptotes with e^x

Exercise

(a) Find all horizontal asymptotes of $f(x) = \dfrac{5e^{3x} - 2}{e^{3x} + 4}$.

(b) A population is modeled by $N(t) = \frac{800}{1+7e^{-0.5t}}$. Find $\lim_{t\to\infty} N(t)$ and interpret it.

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(a) As $x \to \infty$, factor e^{3x} from the top and the bottom:

$$\lim_{x\to\infty} \frac{5e^{3x} - 2}{e^{3x} + 4} = \lim_{x\to\infty} \frac{\cancel{e^{3x}}\left(5 - \frac{2}{e^{3x}}\right)}{\cancel{e^{3x}}\left(1 + \frac{4}{e^{3x}}\right)} = \frac{5 - 0}{1 + 0} = 5$$

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(b) $e^{-0.5t} \to 0$, so $\lim_{t\to\infty} N(t) = \frac{800}{1+0} = 800$: the **carrying capacity**. In the long run the population levels off at 800.

Exercise 4: choose the constants that make f continuous

Exercise

Find a and b so that f is continuous for all x:

$$f(x) = \begin{cases} -2 & \text{if } x \leq 1 \\ ax^2 + b & \text{if } 1 < x \leq 2 \\ 3x + a & \text{if } x > 2 \end{cases}$$

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At $x = 1$: $f(1) = \text{LHL} = \lim_{x \to 1^-}(-2) = -2$

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Need $a + b = -2$.

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$f(2) = \text{LHL} = \lim_{x \to 2^-}(ax^2 + b) = 4a + b$

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Need $4a + b = 6 + a$, i.e. $3a + b = 6$.

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Solve the system

Subtract: $(3a + b) - (a + b) = 6 - (-2)$, so $2a = 8$: $\mathbf{a = 4}$, $b = -2 - 4 = \mathbf{-6}$. (Check: $4(4) - 6 = 10 = 3(2) + 4$.)

Key facts: the derivative and linear approximation (3.3, 3.4, 6.5)

The derivative (3.3, 3.4)

Average ROC on $[a, b]$ (secant slope):

$$\frac{f(b) - f(a)}{b - a}$$

Derivative (tangent slope):

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Starts as $\frac{0}{0}$: simplify until h cancels.

$f'(c)$ **does NOT exist** if f is not continuous at c, or at a corner, cusp, or vertical tangent.

Linear approximation (3.4, 6.5)

Tangent line at $x = a$:

$$y - f(a) = f'(a)(x - a)$$

Solved for y: the linear approximation

$$L(x) = f'(a)(x - a) + f(a)$$

and $f(x) \approx L(x)$ for x NEAR a.

Change: $f(b) - f(a) \approx f'(a)(b - a)$

(> 0: increasing; < 0: decreasing)

Exercise 5: average and instantaneous rates of change

Exercise

A tomato seedling is $H(t) = 6\sqrt{t+1}$ centimeters tall t weeks after it is planted.

(a) Find the average rate of change of H from $t = 0$ to $t = 3$, and interpret it.

(b) Use the limit definition to find $H'(t)$. Then find $H'(3)$ and interpret it.

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$$\frac{H(3) - H(0)}{3 - 0} = \frac{6\sqrt{4} - 6\sqrt{1}}{3} = \frac{12 - 6}{3} = 2 \text{ cm per week}$$

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Over its first 3 weeks, the seedling grows by an *average* of 2 cm per week.

Exercise 5 (b): $H'(t)$ by the limit definition

$$H'(t) = \lim_{h \to 0} \frac{6\sqrt{t+h+1} - 6\sqrt{t+1}}{h} = \frac{0}{0}$$ multiply by the conjugate

Exercise 5 (b): $H'(t)$ by the limit definition

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$$= \lim_{h\to 0} \frac{6\left(\sqrt{t+h+1} - \sqrt{t+1}\right)}{h} \cdot \frac{\sqrt{t+h+1} + \sqrt{t+1}}{\sqrt{t+h+1} + \sqrt{t+1}}$$

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&= \lim_{h\to 0} \frac{6\big[(t+h+1) - (t+1)\big]}{h\left(\sqrt{t+h+1} + \sqrt{t+1}\right)} = \lim_{h\to 0} \frac{6\cancel{h}}{\cancel{h}\left(\sqrt{t+h+1} + \sqrt{t+1}\right)}
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At $t = 3$

$H'(3) = \frac{3}{\sqrt{4}} = \frac{3}{2}$: three weeks after it is planted, the seedling is growing **1.5 cm per week**.

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$$\text{Slope} = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\frac{6}{3+h} - 2}{h} = \frac{0}{0}$$

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Point of tangency and tangent line

$(1, f(1)) = (1, 2)$ and slope $-\frac{2}{3}$: $\qquad y - 2 = -\frac{2}{3}(x - 1)$

Exercise 7: linear approximation of a change

Exercise

The number of people (in hundreds) infected with a flu t days after an outbreak is $P(t) = -t^3 + 12t^2 + 5$ for $0 \leq t \leq 12$.

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Point: $P(3) = -27 + 108 + 5 = 86$, so $(3, 86)$. $L(t) = 45(t - 3) + 86$

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(b) $P(3.5) - P(3) \approx L(3.5) - L(3) = \big[45(0.5) + 86\big] - \big[45(0) + 86\big] = 45(0.5) = 22.5$

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Interpretation

Positive, so **increasing**: about 22.5 hundred = **2,250 more people** are infected from day 3 to day 3.5.

Key facts: the derivative rules (4.1–4.3)

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
constant	0	$\sin(x)$	$\cos(x)$
x^n	$n\,x^{n-1}$	$\cos(x)$	$-\sin(x)$
e^x	e^x	$\tan(x)$	$\sec^2(x)$
$\ln(x)$	$\dfrac{1}{x}$	$\sec(x)$	$\sec(x)\tan(x)$
		$\cot(x)$	$-\csc^2(x)$
		$\csc(x)$	$-\csc(x)\cot(x)$

Rules

$$\big[c\,f(x)\big]' = c\,f'(x)$$

$$\big[u(x) \pm v(x)\big]' = u'(x) \pm v'(x)$$

$$\big[u(x)\,v(x)\big]' = u'(x)\,v(x) + u(x)\,v'(x)$$

$$\left[\frac{u(x)}{v(x)}\right]' = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{\big[v(x)\big]^2}$$

$$\Big[f\big(g(x)\big)\Big]' = f'\big(g(x)\big)\cdot g'(x)$$

Chain rule: derivative of the **outside** evaluated at the **inside**, times the derivative of the **inside**.

Exercise 8: the basic, product, and quotient rules

Exercise

Calculate the derivative.

(a) $f(x) = 5x^4 - \dfrac{3}{x^2} + 4\sqrt{x} - e^3$ (b) $g(x) = x^5 \ln(x)$ (c) $h(x) = \dfrac{e^x}{3x - 2}$

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(a) Rewrite with powers of x first: $f(x) = 5x^4 - 3x^{-2} + 4x^{1/2} - e^3$.

$$f'(x) = 20x^3 - 3(-2)x^{-3} + 4 \cdot \tfrac{1}{2}x^{-1/2} - 0 = 20x^3 + 6x^{-3} + 2x^{-1/2}$$

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Caution: $e^3 \approx 20.09$ is a *constant*, so its derivative is 0 (not $3e^2$, and not e^3).

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(b) Product rule, $u = x^5$ and $v = \ln(x)$: $g'(x) = 5x^4 \cdot \ln(x) + x^5 \cdot \dfrac{1}{x} = 5x^4 \ln(x) + x^4$

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(c) Quotient rule, $u = e^x$ and $v = 3x - 2$: $h'(x) = \dfrac{e^x(3x-2) - e^x \cdot 3}{(3x-2)^2} = \dfrac{e^x(3x-5)}{(3x-2)^2}$

Exercise 9: the chain rule

Exercise

Calculate the derivative. You do not need to simplify.

(a) $f(x) = (3x^2 - 5)^6$ (b) $g(x) = \ln\big(\cos(x)\big)$ (c) $h(x) = x^2 e^{-3x}$

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(a) outside $= x^6 \xrightarrow{\frac{d}{dx}} 6x^5$; inside $= 3x^2 - 5 \xrightarrow{\frac{d}{dx}} 6x$

$$f'(x) = 6(3x^2 - 5)^5 \cdot 6x = 36x(3x^2 - 5)^5$$

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(b) outside $= \ln(x) \xrightarrow{\frac{d}{dx}} \frac{1}{x}$; inside $= \cos(x) \xrightarrow{\frac{d}{dx}} -\sin(x)$

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(c) Product rule; for e^{-3x}: outside $= e^x \xrightarrow{\frac{d}{dx}} e^x$; inside $= -3x \xrightarrow{\frac{d}{dx}} -3$

$h'(x) = 2x\, e^{-3x} + x^2\, e^{-3x} \cdot (-3)$

Key facts: graphs and the derivative (5.1–5.3)

First derivative (5.1, 5.2)

$f' > 0$: f increasing; $f' < 0$: f decreasing.

Critical point (cp): an interior point c of the domain where $f'(c) = 0$ or $f'(c)$ DNE.

First derivative test at a cp: f' goes $\oplus$ to $\ominus$: local max; $\ominus$ to $\oplus$: local min; no sign change: neither.

Second derivative (5.3)

$f'' > 0$: concave up; $f'' < 0$: concave down.

Inflection point: a point *in the domain* where the concavity changes (candidates: $f'' = 0$ or DNE).

Second derivative test: if $f'(c) = 0$, then $f''(c) < 0$: local max; $f''(c) > 0$: local min; $f''(c) = 0$: no conclusion.

Sign charts

Split the number line at the cps *and* at the VAs; test one point in each interval.

Exercise 10: a rational function, start to finish

Exercise

For $f(x) = \dfrac{x^2}{x-1}$, $f'(x) = \dfrac{x(x-2)}{(x-1)^2}$, and $f''(x) = \dfrac{2}{(x-1)^3}$, find

(a) where f is increasing and decreasing, (b) the x-coordinates of all local extrema,

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(e) Check your answer to (b) with the second derivative test.

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Cps of $f(x)$:

- $f'(x) = 0$ (numerator $= 0$): $x(x-2) = 0$, so $x = 0$ and $x = 2$, both in the domain.

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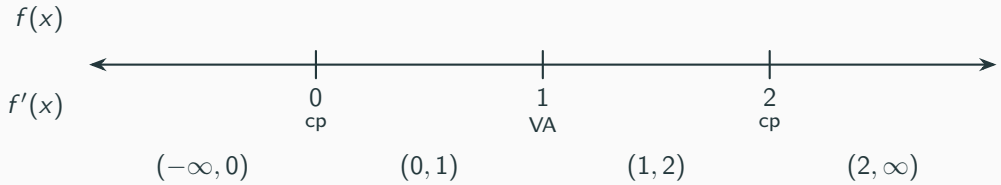
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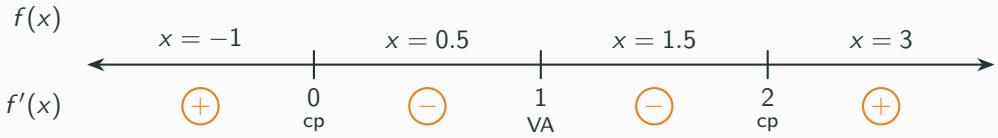
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- $f'(x)$ DNE (denominator $= 0$) at $x = 1$: **NOT** in the domain (a VA), so not a cp.

Number line for f': split at 0 (cp), 1 (VA), 2 (cp): $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, \infty)$.

Exercise 10, continued: sign chart for $f'(x) = \frac{x(x-2)}{(x-1)^2}$

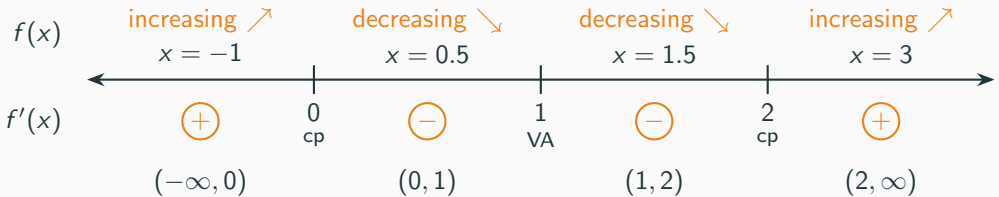


Exercise 10, continued: sign chart for $f'(x) = \frac{x(x-2)}{(x-1)^2}$



The denominator $(x-1)^2$ is positive, so only $x(x-2)$ matters: $f'(-1) = \frac{3}{4}$, $f'(0.5) = -3$, $f'(1.5) = -3$, $f'(3) = \frac{3}{4}$.

Exercise 10, continued: sign chart for $f'(x) = \frac{x(x-2)}{(x-1)^2}$



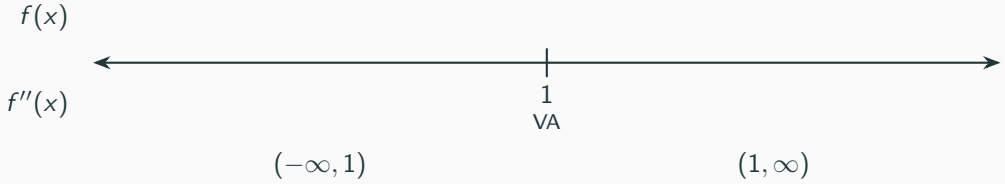
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(a) and (b)

(a) increasing on $(-\infty, 0)$ and $(2, \infty)$; decreasing on $(0, 1)$ and $(1, 2)$.
(b) **local max at** $x = 0$ ($\oplus$ to $\ominus$) and **local min at** $x = 2$ ($\ominus$ to $\oplus$).

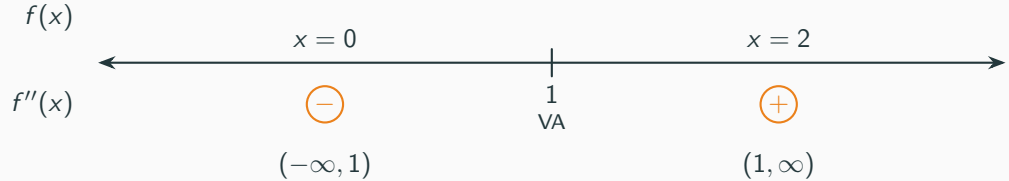
Exercise 10, continued: concavity and the second derivative test

$f''(x) = \frac{2}{(x-1)^3}$ is never 0, and it is undefined only at $x = 1$, which is **NOT** in the domain (VA).



Exercise 10, continued: concavity and the second derivative test

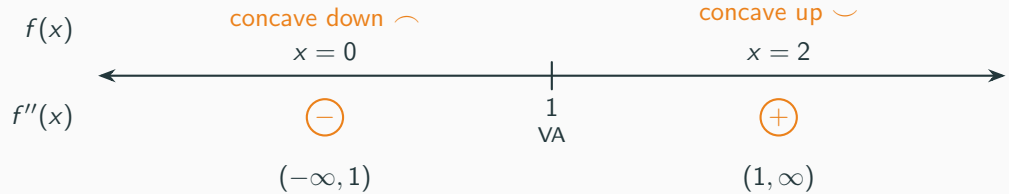
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$f''(0) = \frac{2}{(-1)^3} = -2 \Rightarrow \ominus, \qquad f''(2) = \frac{2}{1^3} = 2 \Rightarrow \oplus$

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(c), (d), and (e)

(c) concave down on $(-\infty, 1)$; concave up on $(1, \infty)$.

(d) **no** inflection points: the concavity changes only across $x = 1$, which is not in the domain.

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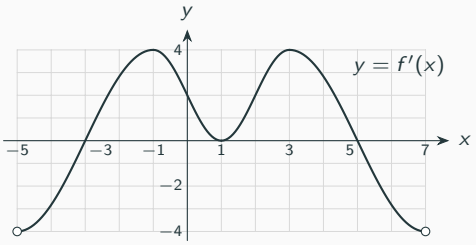
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(e) $f''(0) = -2 < 0$: **local max** at $x = 0$; $f''(2) = 2 > 0$: **local min** at $x = 2$ — as in (b).

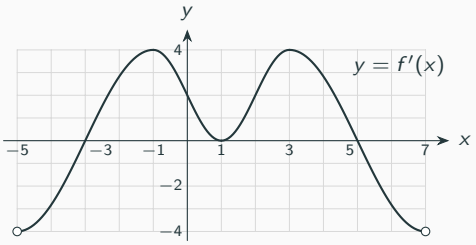
Exercise 11: reading f from the graph of f'

Exercise. Below is the graph of $y = f'(x)$; the domain of f is $(-5, 7)$. Find (a) where f is increasing and decreasing, (b) the x-coordinates of all local extrema, (c) where f is concave up and concave down, (d) the x-coordinates of all inflection points.



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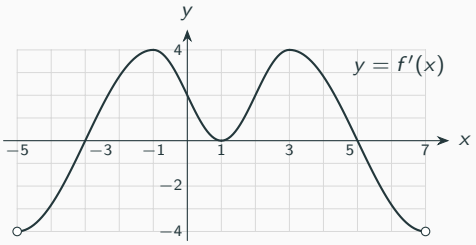
(a), (b): the sign of f'

increasing ($f' > 0$): $(-3, 5)$

decreasing ($f' < 0$): $(-5, -3)$, $(5, 7)$

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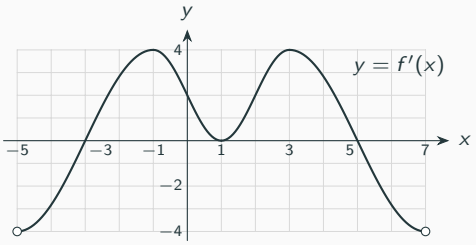


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local min at $x = -3$ ($\ominus$ to $\oplus$)
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none at the cp $x = 1$ (f' stays $\oplus$)

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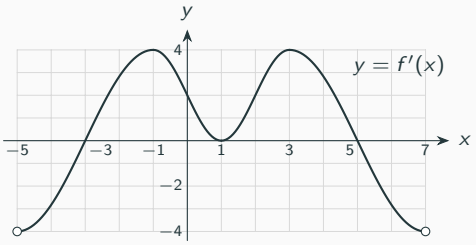
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none at the cp $x = 1$ (f' stays $\oplus$)

(c), (d): is f' rising or falling?

concave up (f' rising): $(-5, -1)$, $(1, 3)$
concave down (f' falling): $(-1, 1)$, $(3, 7)$

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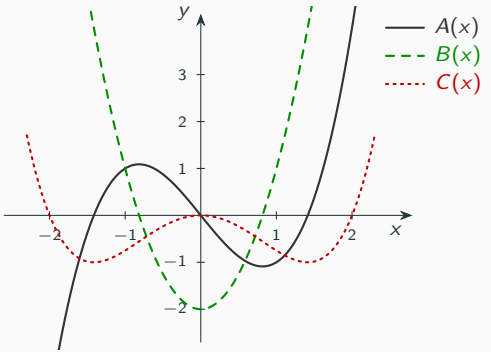
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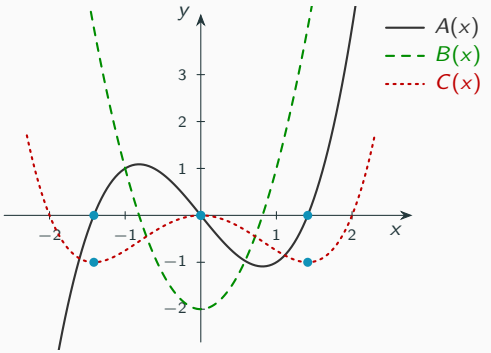
concave up (f' rising): $(-5, -1)$, $(1, 3)$
concave down (f' falling): $(-1, 1)$, $(3, 7)$
inflection points at $x = -1$, 1, 3 (where f' turns)

Exercise 12: which graph is which?



Exercise. The figure shows the graphs of f, f', and f''. Identify each one.

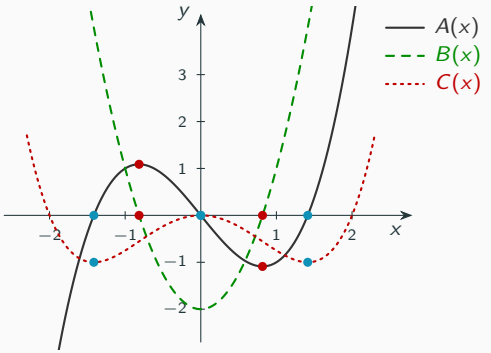
Exercise 12: which graph is which?



Exercise. The figure shows the graphs of f, f', and f''. Identify each one.

1. C has turning points at $x = 0$ and $x \approx \pm 1.4$, and A is **zero** exactly there ($A < 0$ where C falls, $A > 0$ where C rises). So $A = C'$.

Exercise 12: which graph is which?

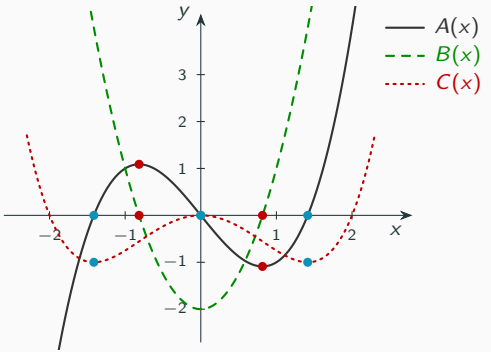


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1. C has turning points at $x = 0$ and $x \approx \pm 1.4$, and A is **zero** exactly there ($A < 0$ where C falls, $A > 0$ where C rises). So $A = C'$.

2. A has turning points at $x \approx \pm 0.8$, and B is **zero** exactly there. So $B = A'$.

Exercise 12: which graph is which?



Exercise. The figure shows the graphs of f, f', and f''. Identify each one.

1. C has turning points at $x = 0$ and $x \approx \pm 1.4$, and A is **zero** exactly there ($A < 0$ where C falls, $A > 0$ where C rises). So $A = C'$.

2. A has turning points at $x \approx \pm 0.8$, and B is **zero** exactly there. So $B = A'$.

Conclusion

$C = f, \quad A = f', \quad B = f''.$